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Valery V. Kozlov
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Asymptotic Solutions of Strongly Nonlinear Systems of Differential Equations

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Translator's Note

The translator has worked with the theory of differential equations and its applications for many years, and yet the translation of the present book presented significant challenges, for from the outset new territory needed to be explored and many new ideas assimilated. At each iteration, in reviewing the translation, one or two subtleties were discovered that caused a change in meaning or emphasis. Hopefully this iterative process has converged to a result that faithfully represents the Russian original, albeit without its eloquence.

The translator has taken the opportunity to prepare an introductory survey lecture for the material of the book that is directed toward colleagues from various disciplines (since much of the stimulus for the range of problems considered in the book comes from the physical sciences) and will welcome opportunities for its dissemination. Questions and corrections for the book can be sent to the dedicated e-mail address: asymptoticsolutions@gmail.com.

Newmarket, NH, USA
Badenweiler, Germany

Lester J. Senechal

Preface

For us, the authors of the monograph *Asymptotic Solutions of Strongly Nonlinear Systems of Differential Equations*—the first Russian edition (1996) by Moscow University Press and the second edition (2009) by the publisher R&C Dynamics (http://shop.rcd.ru)—the decision by Springer-Verlag to publish an English translation of the book is an important event. It is not merely an offspring born from the pangs of creative struggle and thus a favorite, for there are many books, but we present to the Western reader a monograph that is a very special book.

In what way special? First, even at the moment the first Russian edition appeared in 1996, it was the fruit of a decade and a half of research. The first publication of V.V. Kozlov on the subject goes back to 1982 [103, 104, 117] and is dedicated to a very important fundamental problem: the inversion of Lagrange's theorem on the stability of equilibrium—not a trivial task, over which researchers had struggled for more than half a century. The idea behind that 1982 paper belongs to N.G. Chetaev: "To prove the instability of equilibrium is to find just one trajectory of the system that tends to the equilibrium position as time decreases indefinitely". But the word "find" is easily said! The 1982 paper showed that the solution of the equations of motion of a natural mechanical system, with some restrictions, can be constructed in the form of generalized power series whose convergence is proved by quite sophisticated methods of functional analysis.

For a long time our efforts were devoted to this issue and matters related to it. Years passed, and at last it dawned upon us that the method—developed to address a very famous, interesting, and yet particular problem—was in fact universal, giving scientists the opportunity to "view from above" the problem of the asymptotic behavior of differential equations in the neighborhood of a nonelementary singular point. In addition, it was discovered that the technique used in the 1982 paper, which seemed so successful and even elegant, has a deep connection with the fundamental works of such great classics as Lyapunov and Poincaré. This "view from above" which we have perhaps only by the grace of God, gave access to an immense number of applications. Today we scarcely can find an example from any of the previously studied critical cases of the stability of equilibrium of autonomous systems of ordinary differential equations for which sufficient conditions for instability cannot be obtained by the method developed. Moreover, we were able to obtain conditions for instability in some previously unexplored cases and it was found that the method works well for periodic and quasi-periodic time-dependence of systems of differential equations. We also managed to make significant progress on the problem of inverting Lagrange's theorem on the stability of equilibrium and its "little sister": Routh's theorem, for which—as we know by the existence of the phenomenon of gyroscopic stabilization—the converse is simply not true without the imposition of additional conditions. Finally, the method provides enhanced results on the instability of equilibrium for systems with time lags. Work on the method led (as so often happens) to unexpected secondary effects: it turned out that the behavior of solutions in the neighborhood of a trajectory, whose existence is guaranteed by the method, can indicate when systems are nonintegrable or chaotic in one sense or another.

We are absolutely certain that our published work ends not with a period but rather with a comma and that the possibilities that the method offers are far from exhausted. We therefore think of the publication of our monograph by Springer-Verlag as an invitation to Western scholars to continue research in this direction. To the great Russian poet Sergei Yesenin belong the words: "big things can only be seen at a distance". We thus hope that our readers can bring a fresh perspective to the methodology that we have developed and extend the range of problems that can be solved with it.

We are deeply grateful to Springer-Verlag and in particular to Ruth Allewelt, without whose perseverance the preparation time for the book's publication might have become asymptotically infinite, and to the translator, Lester Senechal, whose interest and assistance have helped preserve our hopes for the book's realization.

Moscow, Russia V.V. Kozlov
S.D. Furta

Contents

Introduction

The study of asymptotic properties of solutions of differential equations in the neighborhood of a critical point has almost always accompanied the development of the theory of stability of motion. It was not just as an empty exercise that A.M. Lyapunov, the founder of the classical theory of stability, developed two methods for investigating the behavior of a solution of a system of differential equations in the neighborhood of a critical point. And, if the so-called *second* or *direct Lyapunov method* has mainly a qualitative character and is intended for answering the question "do solutions leave some small neighborhood of a critical point, having begun close to that point?" then *Lyapunov's first method* is dedicated to the analytic representation of solutions in the neighborhood of an equilibrium position. Lyapunov's basic result in this direction, obtained for autonomous systems, consists of the following: if the characteristic equation for the first-order approximation system has s roots with negative real part, then the full system of differential equations has an s-parameter family of solutions beginning in a small neighborhood of the equilibrium solution and converging exponentially toward this solution [133]. In the literature, this result bears the name "Lyapunov's theorem on conditional asymptotic stability". But, as Lyapunov himself remarked [133], this assertion was known yet earlier to Poincaré and was actually contained in Poincaré's doctoral dissertation [151]. There likewise exists a conceptually closely related result, known as the Hadamard-Perron theorem in the literature (although the assertion as it was formulated in the original papers [72, 150] only rather distantly recalls the theorem in its contemporary form): if the characteristic equation of the system of first approximation has s roots with negative real part and p roots with positive real part, then, in the neighborhood of the critical point, there exist two invariant manifolds with respective dimensions s and p, the first consisting of solutions converging exponentially to the critical point as $t \to +\infty$, and the second of solutions converging exponentially to the critical point as $t \to -\infty$. This theorem

The Introduction is substantially a translation of the Preface to the original Russian edition.

and its modern proof can be found in most monographs on differential equations and bifurcation theory [41, 42, 76, 81, 137].

Lyapunov's method is based on asymptotic integration of the system of differential equations being investigated, in the form of certain series containing multiple complex exponentials, the coefficients of which are polynomials in the independent variable. This method was later further developed in the paper [77] and the book [76]. It should be mentioned that, if complex numbers appear among the eigenvalues of the first approximation system, then the construction of real solutions in the form of the aforementioned series is not at all a simple problem, due to the exceedingly awkward appearance of the real solutions, and results in an exorbitant number of computations. Among the classical works devoted to the asymptotics of solutions approaching a critical point as $t \to +\infty$ or $t \to -\infty$, we should also mention the papers of P.G. Bol' [23].

In many concrete problems, the fundamental method of proof of stability is the method for constructing Chetaev functions [39, 40], which make it possible to give a qualitative picture of the behavior of trajectories in a whole domain of phase space bordering the critical point. This method is in a certain sense "excessive", inasmuch as—as was noted by Chetayev himself—"in order to reveal the instability of unperturbed motion, it suffices to observe, among all trajectories, just one that exits from a given region under perturbations of arbitrarily small numerical value" [39, 40]. It was observed long ago that in the majority of cases the condition of instability of a critical point of a system of ordinary differential equations is accompanied by the existence of a particular solution of the system, converging to this point as $t \to -\infty$. But the most general conditions for Chetaev functions, under which asymptotic solutions actually exist, were found considerably later by N.N. Krasovskiy [124]. In problems related to the generalized Lagrange stability theorem, this phenomenon was noted rather long ago. It is worth stressing that the reversibility of the equations of motion of a conservative mechanical system guarantees simultaneously the existence and "entering" of solutions, i.e., the convergence to the equilibrium state as $t \to +\infty$. One of the first papers where this phenomenon is described is due to A. Kneser [101]. It was later the subject of generalization by Bohl [22]. In a paper by our first author [103], requirements on Chetayev functions of a special type were presented and used in the proof of instability of equilibrium states of reversible conservative systems, ensuring the existence of "exiting" solutions.

Practically all the results cited above concern cases where the presence of an asymptotic solution, tending to a critical point during unbounded increase or decrease in the independent variable, is successfully detected based only on an analysis of the linearized equation in cases where the convergence of solutions to critical points displays an exponential character. But the detection of solutions whose convergence to a critical point is nonexponential, and likewise the construction of asymptotics for such solutions, represents a more difficult problem. It is all the more paradoxical that one of the first papers dedicated to a "nonexponential" problem was published long before the appearance of Lyapunov's "General Problem of the Stability of Motion." We are referring to the work of Briot and Bouquet [28], which served as a basis of the development by G.V. Kamenkov [93] of a method for constructing invariant curves, along which a solution of nonexponential type departs from a critical point. For many years, this method remained a reliable technique in mechanics for proving instability in so-called critical cases, based only on linearized equations (see, e.g., the monograph of V.G. Veretennikov [188]). The work of Kamenkov just mentioned is almost entirely unknown to nonmathematicians, for the unfortunate reason that the only available publication containing the formulation and proof of Kamenkov's theorem is the posthumous collection of his papers already cited above [93]. Although the formulation of the theorem was absolutely correct, the proof as presented contained some technical assertions that were regarded as obvious when they really requiredadditional analysis. Among the more contemporary authors who established the beginnings of research into the asymptotics of solutions of systems of differential equations in the neighborhood of nonelementary critical points, we should mention A.A. Shestakov in connection with his paper [164]. One of the first papers which discussed the possibility of constructing asymptotics of solutions to differential equations in power form was the paper of N.V. Bugaev [33]. A.D. Bryuno [29] proposed a general algorithm, based on the technique of the Newton polytope, for computing the leading terms of expansions of solutions, possessing a generalized power asymptotic, of an analytic system of differential equations in the neighborhood of a nonelementary critical point. Newton diagrams and polytopes play an important role in various mathematical areas. At the "top of the list" of our national literature in applying these fascinating techniques to contemporary research, it is a pleasure to cite the paper on Newton's polygon by N.G. Chebotarev [38].

Newly increasing interest toward the end of the 1970s in the inverse Lyapunov problem on stability gave new impetus to the investigation of asymptotics for solutions of differential equations in the neighborhood of an elementary critical point. In a paper by our first author [104], and likewise in the paper [117] written by him incollaboration with V.P. Palamodov, asymptotic solutions of the corresponding equations of motion were found in the form of various series in the quantity $t^{-\alpha j} \ln^k t$, which in form coincide with expansions in just one of the variables x_j that had been proposed by G.V. Kamenkov [93]. It should be remarked that the appearance of logarithmic terms in asymptotic expansions of solutions to nonlinear equations is a very general phenomenon (see, e.g., [15]). In this connection, it was discovered that in a whole multitude of cases the constructed series can diverge, even if the right sides of the investigated equations are holomorphic in the neighborhood of the critical point [106]. The way out of this dilemma is to apply the theory of A.N. Kuznetsov [125, 126], which establishes a correspondence between the formal solutions of the nonlinear system of equations being investigated and certain smooth particular solutions having the required asymptotic. Application of this technique allowed our second author to give an elementary and rigorous proof of Kamenkov's theorem [60].

The question arises as to whether the leading terms of the expansions of nonexponential asymptotic solutions always have the form of a power. As follows from the paper of A.P. Markeev [136], which deals with the existence of asymptotic trajectories of Hamiltonian systems in critical cases, the answer to this question is negative: for example, in the presence of fourth-order resonances between frequencies of the linearized system, asymptotic solutions entering the critical point may have a more complicated form than powers. Other examples of this type can be found in one of the articles by Kuznetsov cited above [126].

In the last 10 or 15 years, the problem of existence of particular solutions of systems of differential equations with nonexponential asymptotic has attracted the attention of theoretical physicists. The situation is that the structure of these solutions is closely connected with the Painlevé property [68, 87]. By the "Painlevé property" in the extensive literature by physicist-geometers (see, e.g., the survey [26]) is meant the following: (a) movable singularities of solutions in the complex domain can only be poles and (b) the formal expansions of these solutions into Laurent series contain $(n - 1)$ arbitrary constants as free parameters, where n is the dimension of the phase space. The test of these properties bears the name "Painlevé test" or ARS-test after the names of the authors who were among the first to apply the stated approach to nonlinear problems of mathematical physics [1]. Practically all the systems that pass the ARS-test can be integrated in explicit form [1, 26, 49]. The idea that the solutions of the integrated system must be single-valued meromorphic functions of time goes back to Kovalevsky [102]. It seems a plausible hypothesis that nonintegrable systems cannot satisfy the Painlevé property. However, there are but few rigorous results on nonintegrability that use nonexponential asymptotics. In this connection, it is worthwhile recalling the work of H. Yoshida [197] (where there are some inaccuracies that have been pointed out in [69]) and also a series of rather recent articles, also by Yoshida [198–200], based on a method of S.L. Ziglin [202], which include much stronger conditions on the system considered. It has been observed that the presence of logarithmic terms in the asymptotic expansions of solutions of many concrete systems, ordinarily considered chaotic, actually corresponds to very intricate behavior of trajectories and comes down to the fact that the singularities of these solutions form capricious star-shaped structures in the complex plane that resemble fractals [52, 181]. Very similar effects are engendered by the presence of irrational and complex powers in the asymptotic expansions of solutions [36, 37].

The purpose of our monograph is to systematically set forth the present state of affairs in the problem of investigating the asymptotics of solutions of differential equations in the neighborhood of a nonelementary singular point, to indicate ways of extending this theory to other objects of a dynamical nature and likewise to demonstrate the wide spectrum of applications to mechanics and other fields. In all this the authors make no pretense toward a complete bibliographical survey of work related to our problem. Many of the results indicated in the book have been obtained by the authors themselves, so that the material presented is fundamentally determined by their viewpoints and biases.

Before beginning a brief account of results for our readers' attention, it is necessary to make a remark of a bibliographical nature. At first glance, it may seem that problems of constructing particular solutions of differential equations with exponential or generalized power asymptotics differ so much from one another that, in their construction, we could scarcely succeed in using some idea encompassed in the first Lyapunov method. This misunderstanding becomes attenuated by a cursory acquaintance with these objects. Wherein lies the essence of Lyapunov's first method? In his famous "General Problem of the Stability of Motion", [133] Lyapunov, while considering systems of ordinary differential equations of the form

$$\dot{x} = \mathbf{f}(\mathbf{x}),\ \mathbf{f}(\mathbf{0}) = \mathbf{0},\ \mathbf{x} \in \mathbb{R}^{\mathbf{n}},\ \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \ldots,$$

the dots representing the totality of the nonlinear terms, proposed looking for particular solutions in the form of a series

$$\mathbf{x}(t) = \sum_{j_1+\ldots+j_p\geq 1}^{\infty} \mathbf{x}_{j_1,\ldots,j_p}(t)\exp((j_1\lambda_1 + \ldots + j_p\lambda_p)t),\ p \leq n,$$

where the functions $x_{j_1,\ldots,j_p}(t)$ depend polynomially on t and, if we wish to limit ourselves to consideration of real solutions, on some trigonometric functions of time.

In all this, it is supposed that the totality of the first terms of theseseries (i.e., where $j_1 + \ldots + j_p = 1$) is a solution of the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

In this way, the application of Lyapunov's first method demands three steps:

1. Selection from the system of some truncated (in the present instance linear) subsystem
2. Construction of a particular solution (or a family of particular solutions) of the given truncated system
3. Adding to the found supporting solution (or family of supportingsolutions) to get the solution of the full system with the aid of some series

This same scheme is also used for constructing particular solutions with generalized power asymptotic. Throughout the reading of the book, other deeper analogies between these two problems will come to light. The more essential results presented herein were published in the article [116].

We likewise turn the attention of readers to the term *strongly nonlinear system*, used in the title, which could raise a number of questions. If none of the eigenvalues of the system, linearized in the neighborhood of a critical point, lies on the imaginary axis, then this system is topologically conjugate to its linear part (Grobman-Hartman theorem) [7, 76]. From this point of view, it is natural to call a system strongly nonlinear if the topological type of its phase portrait in the neighborhood of the critical point is not completely determined by its linear terms. We therefore include in this class any system for which the characteristic equation of its linear part has a pure imaginary or zero root.

This book consists of four chapters and two appendices. The first chapter is dedicated to the theory of so-called *semi-quasihomogeneous systems*, i.e., those that are "almost" invariant with respect to the action of the phase flow of a certain special linear system of equations of Fuchsian type, and the construction of their solutions. The action of this flow introduces a certain small parameter that allows us to select a truncated quasihomogeneous subsystem. In the first section of this chapter, the fundamental theorem is proved that particular solutions of the truncated system lying on the orbits of the indicated flow can be fully constructed with the help of certain series for the solutions of the full system. The second section is dedicated to an analysis of the convergence of those series. As has already been noted, the problem of finding solutions of a system of differential equations with generalized power asymptotic has its roots in Lyapunov's first method. The third section is dedicated to applying the idea of this method to the problem considered. In particular, with the aid of the "quasilinear" technique, a theorem is proved on the existence of multiparameter families of solutions with nonexponential asymptotic for systems of equations of a much broader class than had been considered in the preceding section. The subsequent fourth section contains a large collection of concrete examples from mathematics, mechanics, physics, and other branches of natural science. On the one hand, these examples illustrate the proved theoretical results, and on the other, they have an independent significance. For example, we investigate a new critical case of high codimension n of zero roots of the characteristic equation with one group of solutions. Another interesting application consists of methods for constructing collision trajectories in real time for the Hill problem. In the fifth and final section, we discuss a group theoretical approaches to the problem of constructing particular solutions of systems of differential equations. The proposed method is based on using arbitrary one-parameter groups of transformations of phase space, being in some sense "almost" a symmetry group for the system of equations considered.

The second chapter is dedicated to finding sufficient conditions for the existence of solutions of systems to differential equations that converge to a critical point as $t \to +\infty$ or $t \to -\infty$ when the first approximation system is neutral. In Sect. 2.1 we introduce sufficient conditions for the existence of such solutions for truncations of the Poincaré normal form. The critical case of two pure imaginary roots for a general four-dimensional system of differential equations is considered at length. In Sect. 2.2, the results obtained in Sect. 2.1 are generalized to systems for which the right sides depend periodically on time. An analogous theory is constructed for problems of finding solutions that converge asymptotically to invariant tori, provided these tori are neutral in the first approximation. In Sect. 2.3, we discuss characteristics of solutions of problems considered in the first section induced by the Hamiltonian property of the system considered. Here it is shown by way of illustration precisely how, with the help of the stated results, we obtain known theorems on the instability of the equilibrium position of Hamiltonian systems with two degrees of freedom in the presence of resonances between frequencies of small vibrations.

In the third chapter, problems are considered that the authors call singular. The peculiarity of these problems lies in the fact that series that represent solutions, and have the required asymptotic, diverge even in the case where the system under investigation is analytic. In the first section, dedicated to a method of obtaining enough instability conditions in the critical case where there is at least one zero root of the first approximation system, we prove a theorem on the existence of a formal invariant manifold, to which the linear subspace corresponding tosuch zero roots is tangent. It is shown that nonanalyticity for this manifold is inevitable and is the reason for the divergence of the aforementioned series, whose asymptotic expansions contain both exponentials and negative powers of the independent variable. It was noted long ago that an asymptotic solution of a system of differential equations can contain powers of iterated logarithms of rather high orders. The mechanism of this phenomenon is revealed in Sect. 3.2. It is also connected with certain "critical" cases, where there is a zero present in the spectrum of the so-called Kovalevsky matrix. In Sect. 3.2, another reason for the divergence of the asymptotic series is set forth, which amounts to the fact that, in selecting a quasihomogeneous truncation, even with the aid of the standard methods of Newton polytopes, certain derivatives may vanish. We introduce several theorems on the existence of asymptotic solutions of systems which are implicit (for systems in which only first derivatives occur, explicit means that only individual first-order derivatives appear on the left-hand side) in the derivatives. We briefly set forth the above-mentioned theory of Kuznetsov [125, 126], allowing us to set up a correspondence between formal and actual solutions of such systems. By way of an example with critical positions, we discuss a series of papers dedicated to new integrable cases in the problem of the motion of a massive solid body about a fixed point with the aid of Kovalevsky's method. It is shown that the solutions obtained in these papers are above all *not analytic*, since, for their construction, the authors had to use the truncation procedure for an Euler-Poisson system, leading to loss in differentiability.

The material presented in Chaps. 2 and 3, and in a part of Chap. 1, is first of all a powerful means for proving the instability of a critical point for a system of differential equations. Therefore the majority of theorems concerning the existence of asymptotic solutions are accompanied by dual theorems that give sufficient conditions for instability.

The fourth chapter has an illustrative character. In it we consider a range of problems that in one way or another are connected with the converse to Lagrange's theorem on the stability of equilibrium. In Sect. 4.1, we present (basically without proof) sufficient conditions for stability (including asymptotic stability) of an equilibrium position of generalized gradient systems, for reversible conservative mechanical systems, for mechanical systems on which a dissipative gyroscopic force is acting, and, likewise, for systems whose parameters change with time. These conditions are expressed in terms of the presence or absence of a minimum in potential energy at the equilibrium position considered. We likewise consider the problem of imposing some added constraints on the stability of equilibrium. In particular, we give an instructive example that shows that the stabilization in the first approximation of a reversible conservative system by means of imposition of nonholonomic constraints may be either stable or unstable depending on the arithmetic properties of its frequencies. The following two sections are dedicated to the converse of the theorem stated in the first section, with the aid of construction of asymptotic solutions. Their division bears a purely conditional character. If in Sect. 4.2 we use assertions of a "regular" character, i.e., proofs from Chap. 1 that guarantee convergence of the constructed series in the most important applications, then Sect. 4.3 is based on "singular" methods, leading to the construction of divergent series.

Appendix A is dedicated to the extension of well-developed methods for certain other objects of a dynamical character. In this appendix, for systems of differential equations with a deflecting argument and likewise for systems of integro-differential equations of a specific form, we introduce the concepts of quasihomogeneity and semi-quasihomogeneity. Conditions are indicated that are sufficient for the existence of solutions that tend to equilibrium with an unbounded decrease in time and, based on these, instability theorems are formulated. By way of an example, we discuss the interesting effect of explosive instability in ecological systems of the Volterra-Lotka type at zero values of the Malthusian birth rates.

In Appendix B, we consider the problem of how the presence and structure of particular solutions with generalized power asymptotic influences the integrability of systems of ordinary differential equations. We mentioned above the paper of H. Yoshida [197] where, by way of criteria for integrability, the arithmetic properties of the eigenvalues of the Kovalevsky matrix were used, whose calculation is impossible without finding the principal terms of the asymptotic of the particular solutions of nonexponential type. In Appendix B, Yoshida's criteria are sharpened and the result is applied to some systems of equations that are well-known in mathematical physics.

The book is aimed in the first instance at a wide circle of professional scholars and at those who are preparing to be such: pure and applied mathematicians, as well as students, who are interested in problems connected with ordinary differential equations, and specialists in theoretical mechanics who are occupied with questions about the behavior of trajectories of mechanical systems. The authors are also hopeful that any physicist who is attracted by theoretical research will find much that is useful in the book. In order to extend the circle of readers, the authors have attempted to plan the exposition as much as they could in such a way to make the book accessible to readers whose background includes just the standard program of higher mathematics and theoretical mechanics in the applied mathematics department of a technical university. The proofs of theorems on the convergence of formal series, or of the asymptotic behavior of solutions, containing (not at all complicated) elements of functional analysis, are written in such a way that they may be omitted without detriment to the understanding of the basic circle of ideas. We have included in the text information (without proofs) from the theory of normal forms so as not to interrupt the flow of the exposition. In a few places (and in particular in Sect. 1.5 of Chap. 1, Sect. 2.3 of Chap. 2, and Appendix B), the reader will need some elementary material from algebra (e.g., the concepts of group and

Lie algebra). Nonetheless, a series of indispensable assertions concerning groups of symmetries of systems of differential equations are formulated and proved in the text. Some information from differential topology is used, but only in the proof of a single technical lemma and thus it too will not interrupt the ideal simplicity of the exposition.

The present book represents the fruit of more than two decades of work by the authors: their first publications on this subject go back to 1982. This book contains not only work that was published previously, but also results that appear here for the first time. A portion of the material is based on the specialized courses "asymptotic methods in mechanics", given by the first author in the mechanical-mathematical faculty of Moscow State University, and "analytical methods and celestial mechanics in the dynamics of complex objects", given by the second author in collaboration with P.S. Krasilnikov in the faculty of applied mathematics and physics at the Moscow Aviation Institute. A significant portion of the book is also based on mini-courses given by the second author at the Catholic University of Louvain (Belgium) in 1994 and at the University of Trento (Italy) in 1995.

In the process of working on the book, the authors had frequent opportunity for discussing Lyapunov's first method and its application to strongly nonlinear systems with many scholars. We wish to take this opportunity to thank those who participated in these discussions and listened to the nontraditional and perhaps controversial views of the authors on this subject: P. Hagedorn (Technical University Darmstadt), J. Mawhin and K. Peiffer (Catholic University of Louvain), L. Salvadori (University of Trento), and V.V. Rumyantsev and S.V. Bolotin (Moscow State University). We especially wish to thank Professor L. Salvadori for bringing to the authors' attention a range of problems that were previously unknown to them and whose hospitality helped bring the work on this book nearer to completion. We cannot neglect to mention the dedicated work of M.V. Matveev in carefully reading the manuscript and in making comments that helped the authors eliminate a number of errors. Thanks are also due to the book's translator (into English), L.J. Senechal, who has assumed full responsibility for the integrity of the translated version and who performed the translation task in the shortest possible time.

Chapter 1
Semi-quasihomogeneous Systems of Differential Equations

1.1 Formal Asymptotic Particular Solutions of Semi-quasihomogeneous Systems of Differential Equations

We consider an infinitely smooth system of differential equations for which the origin $\mathbf{x} = 0$ is a critical point:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(\mathbf{0}) = \mathbf{0}, \ \mathbf{x} \in \mathbb{R}^n, \tag{1.1}$$

Let $\mathbf{A} = d\mathbf{f}(\mathbf{0})$ be the Jacobian matrix of the vector field $\mathbf{f}(\mathbf{x})$, computed at the critical point $\mathbf{x} = \mathbf{0}$. We will look for conditions on the right side of (1.1) that are necessary in order that there be a particular solution $\mathbf{x}$ such that $\mathbf{x}(t) \to 0$ either as $t \to +\infty$ or as $t \to -\infty$. In the sequel such solutions will be called *asymptotic*.

We first formulate a well known result. Consider the operator $\mathbf{A}\colon \mathbb{R}^n \to \mathbb{R}^n$. We decompose the space $\mathbb{R}^n$ into the direct sum of three spaces

$$\mathbb{R}^n = E^{(s)} \oplus E^{(u)} \oplus E^{(c)}$$

(where s connotes *stable*, u—*unstable*, c—*center*). This decomposition is dictated by the following requirements: all three subspaces on the right hand side are invariant under the operator $\mathbf{A}$; the spectrum of the restricted operator $\mathbf{A}|_{E^{(s)}}$ lies in the left half plane, that of $\mathbf{A}|_{E^{(u)}}$—in the right half plane, and that of $\mathbf{A}|_{E^{(c)}}$—on the imaginary axis. The possibility of such a decomposition follows from standard theorems of linear algebra (see e.g. [74]).

Theorem 1.1.1. *To system (1.1) there correspond three smooth invariant manifolds $W^{(s)}$, $W^{(u)}$, $W^{(c)}$ passing through $\mathbf{x} = \mathbf{0}$ and tangent there to $E^{(s)}$, $E^{(u)}$, $E^{(c)}$ respectively and having the same respective dimensions. The solutions with initial conditions on $W^{(s)}$ $(W^{(u)})$ converge exponentially to $\mathbf{x} = \mathbf{0}$ as $t \to +\infty$ $(t \to -\infty)$, but the behavior of solutions on $W^{(c)}$ is determined by nonlinear elements.*

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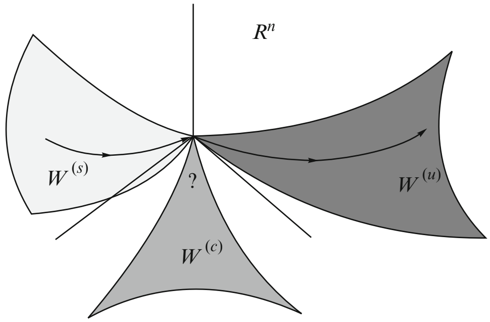

Fig. 1.1 Stable, center and unstable manifolds

The manifold $W^{(s)}$ is called stable, $W^{(u)}$—unstable, and $W^{(c)}$—center (see Fig. 1.1).

The stated theorem is a combination of the Hadamard-Perron theorem and the center manifold theorem [41, 76, 81, 137]. It should be noted that the center manifold $W^{(c)}$ generally has but a finite order of smoothness [41].

Thus the question of the existence of asymptotic particular solutions of (1.1) with exponential asymptotic is solved by studying the first approximation system. We note yet another fact, arising from the general philosophy of Lyapunov's first method: if the matrix $\mathbf{A}$ has at least one nonzero real eigenvalue β, then there exists a real particular solution (1.1), belonging to $W^{(s)}$ or $W^{(u)}$, according to the sign of β, represented in the form

$$\mathbf{x}(t) = e^{-\beta t} \sum_{k=0}^{\infty} \mathbf{x}_k(t) e^{-k\beta t}, \tag{1.2}$$

where the $\mathbf{x}_k(t)$ are certain polynomial functions of the time t and where $\mathbf{x}_0 \equiv$ **const** is an eigenvector of the matrix $\mathbf{A}$ with eigenvalue β. For complex eigenvalues the corresponding decomposition of the real parts of solutions has a much more complicated appearance.

In order to find nonexponential asymptotic solutions, it is necessary to reduce the system on the center manifold. But in this chapter we will assume that the critical case of stability holds both in the future and in the past, i.e. that all eigenvalues of the Jacobian matrix $d\mathbf{f}(0)$ have purely imaginary values; and in this section we will implicitly assume that the operator $d\mathbf{f}(0)$ is nilpotent. Our main task will be to find sufficient conditions for the system of equations (1.1) to have a nonexponential particular solution $\mathbf{x}(t) \to \mathbf{0}$, either as $t \to +\infty$ or as $t \to -\infty$.

In this section we will settle an even more general question. Consider the nonautonomous smooth system of differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t). \tag{1.3}$$

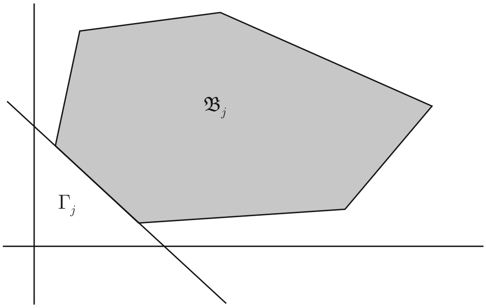

Fig. 1.2 The j-th Newton polytope

and let the vector field $\mathbf{f}(\mathbf{x}, t)$ be such that its components $f^1, \dots, f^n$ can be represented as formal power series

$$f^i = \sum_{i_1,\dots,i_n,i_{n+1}} f^j_{i_1,\dots,i_n,i_{n+1}} (x^1)^i_1 \dots (x^n)^i_n t^{i_{n+1}}, \tag{1.4}$$

where the indices $i_1, \dots, i_n$ are nonnegative integers and i_{n+1} is an integer. Thus we waive the requirement $\mathbf{f}(\mathbf{0}, t) \equiv \mathbf{0}$. For systems of this sort we will find conditions sufficient to guarantee the existence of particular solutions whose components have a generalized power asymptotic either as $t \to \pm 0$ or as $t \to \pm\infty$. Here the smoothness requirements on the right hand terms of (1.3)—even over the Cartesian product of some small neighborhood of $\mathbf{x} = \mathbf{0}$ with a set of the form $0 < \underline{t} < |t| < \overline{t} < +\infty$—may turn out to be inadequate. For this reason we will always assume that the right hand members are infinitely differentiable vector functions over that region of space where the desired solution should be found.

A critical point of a system of differential equations is called *elementary* [32] if the Jacobian matrix of the right hand elements of this system, computed at the critical point, has at least one nonzero eigenvalue. We mention some definitions that are used in the theory of stability of systems with nonelementary critical points [32, 100].

Definition 1.1.1. Let $f^j_{i_1,\dots,i_n,i_{n+1}} (x^1)^i_1 \dots (x^n)^i_n t^{i_{n+1}}$ be some nontrivial monomial in the expansion (1.4), the j-th component of the nonautonomous vector field $\mathbf{f}(\mathbf{x},t)$. We consider in $\mathbb{R}^{n+1}$ a geometric point with coordinates $(i_1, \dots, i_n, i_{n+1})$. The collection of all such points is called the j-th *Newton diagram* $\mathfrak{D}_j$ of the vector field $\mathbf{f}(\mathbf{x},t)$, and its convex hull the j-th *Newton polytope* $\mathfrak{B}_j$ (see Fig. 1.2).

Definition 1.1.2. The vector field $\mathbf{f} = \mathbf{f}_{\mathrm{q}}(\mathbf{x},t)$ is called *quasihomogeneous* of degree $\mathrm{q} \in \mathbb{N}$, $\mathrm{q} \neq 1$, with exponents $s_1, \dots, s_n \in Z$, where the numbers $\mathrm{q} - 1, s_1, \dots, s_n$ don't have any nontrivial common divisor, if for arbitrary $(\mathbf{x}, t) \in \mathbb{R}^{n+1}$, $t \neq 0$, $\lambda \in \mathbb{R}^+$ the following condition is satisfied:

$$f^j_{\mathrm{q}}(\lambda^{s_1} x^1, \dots, \lambda^{s_n} x^n, \lambda^{1-\mathrm{q}} t) = \lambda^{\mathrm{q}+s_j-1} f^j_{\mathrm{q}}(x^1, \dots, x^n, t).$$

It is useful to note that the Newton diagram $\mathfrak{D}_j$ and the Newton polytopes $\mathfrak{B}_j$ of a quasihomogeneous vector field lie on hyperplanes determined by the equations

$$s_1 i_1 + \ldots + s_j(i_j - 1) + \ldots + s_n i_n + (1 - \mathrm{q})(i_{n+1} + 1) = 0. \tag{1.5}$$

The requirement $\mathrm{q} > 1$ is generally redundant: if we examine an individual vector field, then from the sign changes of the quantities $s_1, \ldots, s_n$ we can ascertain that the inequality $\mathrm{q} > 1$ is satisfied.

Definition 1.1.3. Let Γ be some r-dimensional face of the j-th Newton polytope $\mathfrak{B}_j$ $(0 \leq r < n + 1)$, lying in the hyperplane given by Eq. (1.5). The face Γ_j will be called *positive* if an arbitrary point $(i_1, \ldots, i_n, i_{n+1}) \in \mathfrak{B}_j \setminus \Gamma_j$ lies in the positive half-space determined by this hyperplane (see Fig. 1.2), i.e. if it satisfies the inequality

$$s_1 i_1 + \cdots + s_j(i_j - 1) + \cdots + s_n i_n + (1 - \mathrm{q})(i_{n+1} + 1) > 0.$$

Conversely, if for each point in $\mathfrak{B}_j \setminus \Gamma_j$ the opposite inequality

$$s_1 i_1 + \cdots + s_j(i_j - 1) + \cdots + s_n i_n + (1 - \mathrm{q})(i_{n+1} + 1) < 0,$$

holds, then the face Γ_j is said to be *negative*.

Definition 1.1.4. We say that the vector field $\mathbf{f}(\mathbf{x}, t)$ is *semi-quasihomogeneous* if it can be represented in the form

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t) + \mathbf{f}^{*}(\mathbf{x}, t),$$

where $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is some quasihomogeneous vector field determined by all positive, or all negative, faces in the Newton polytope of the total system, and where the exponents of the "perturbing" field $\mathbf{f}^{*}(\mathbf{x}, t)$ lie strictly in the interior of these polytopes (see Fig. 1.2). We also say that the vector field under consideration is *positive semi-quasihomogeneous* if its quasihomogeneous truncation $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is chosen with positive faces; in the opposite case we call the considered field *negative semi-quasihomogeneous*.

In the "positive" case we look for an asymptotic solution of system (1.1) as $t \to \pm\infty$, and in the "negative" case as $t \to \pm 0$.

We introduce the following notation. Let $\mathbf{S}$ be some diagonal matrix

$$\mathrm{diag}(s_i, \ldots, s_n)$$

with integer entries and λ some real number. The symbol $\lambda^{\mathbf{S}}$ denotes the diagonal matrix $\mathrm{diag}(\lambda^{s_1}, \ldots \lambda^{s_n})$.

It is clear that Definition 1.1.2 of the quasihomogeneous vector field $\mathbf{f}_q(\mathbf{x}, t)$ can be written in the following equivalent form: for arbitrary $(\mathbf{x}, t) \in \mathbb{R}^{n+1}$, $t \neq 0$ and arbitrary real λ the following equation must be satisfied:

$$\mathbf{f}_q(\lambda^{\mathbf{S}}\mathbf{x}, \lambda^{1-q}t) = \lambda^{\mathbf{S}+(q-1)\mathbf{E}}\mathbf{f}_q(\mathbf{x}, t), \tag{1.6}$$

where $\mathbf{E}$ is the identity matrix.

The simplest example of a quasihomogeneous vector field is perhaps the homogeneous vector field where $\mathbf{S} = \mathbf{E}$. We will consider other examples of quasihomogeneous vector fields a bit later.

From (1.6) it follows that a quasihomogeneous system of differential equations, i.e. one for which the right side is a quasihomogeneous vector field that is invariant under the quasihomogeneous group of dilations

$$t \mapsto \mu^{-1}t, \ \mathbf{x} \mapsto \mu^{\mathbf{G}}\mathbf{x}, \ \text{where}\mathbf{G} = \alpha\mathbf{S}, \ \alpha = \frac{1}{q-1}. \tag{1.7}$$

It is likewise easy to see that if a system of differential equations is semi-quasihomogeneous—i.e. if its right side is a semi-quasihomogeneous vector field—then under the action of the group (1.7) it will assume the form

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}, t) + \mathbf{f}^*(\mathbf{x}, t, \mu). \tag{1.8}$$

Here $\mathbf{f}_q(\mathbf{x}, t)$ is a quasihomogeneous vector field, chosen with positive or with negative faces for the Newton polytopes, and $\mathbf{f}^*(\mathbf{x}, t, \mu)$ represents a formal power series in μ^β, $\beta \in \mathbb{R}\backslash\{0\}$, $\alpha = |\beta|$ with zero free term. If $\beta > 0$, then $\mathbf{f}(\mathbf{x}, t)$ is positive semi-quasihomogeneous, and if $\beta < 0$ the vector field considered is negative semi-quasihomogeneous.

Setting $\mu = 0$ in (1.8) in the positive semi-quasihomogeneous case and $\mu = \infty$ in the negative semi-quasihomogeneous case, we obtain a "*truncated*" or a "*model*" system, as it is also called:

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}, t). \tag{1.9}$$

We note that the reasoning introduced above can be used for the determination of quasihomogeneous and semi-quasihomogeneous systems. If we aren't interested in the specific numerical values of the powers of a quasihomogeneous system, then its qualitative structure is completely determined by the matrix of the transformation $\mathbf{G}$.

In essence, in order to know the parameters of a semi-quasihomogeneous system, it is sufficient to determine $\beta \neq 0$ and $\mathbf{G}$. For this, it is even possible to avoid the rather burdensome requirement $q \neq 1$. In the sequel we will as before designate the chosen quasihomogeneous truncation by $\mathbf{f}_q$, whereby q signifies the *quasihomogeneity* property rather than the homogeneity degree.

We now introduce a more general definition of quasihomogeneity and semi-quasihomogeneity of vector fields that allows us to avoid both the concept of the degree of quasihomogeneity and the use of techniques that are associated with Newton polytopes.

We consider an $(n + 1)$–dimensional *Fuchsian* system of differential equations of the form

$$\mu \frac{d\mathbf{x}}{d\mu} = \mathbf{G}\mathbf{x}, \quad \mu \frac{dt}{d\mu} = -t, \tag{1.10}$$

where **G** is some real matrix and where the flow will be denoted by

$$t \mapsto \mu^{-1} t, \quad \mapsto \mu^{\mathbf{G}} \mathbf{x}. \tag{1.11}$$

We recall that the right hand side of a linear Fuchsian system has singularities in the form of simple poles in the independent variable (in the present instance, in μ).

Definition 1.1.5. The vector field $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is quasihomogeneous if the corresponding system of differential equations is invariant with respect to the action of the phase flow (1.11) of the Fuchsian system (1.10).

This definition is equivalent to Definition 1.1.2 if we set

$$\mathbf{G} = \alpha \mathbf{S}, \ \lambda = \mu^{\alpha}, \ \textbf{where } \alpha = 1/(\mathrm{q} - 1).$$

The definition of semi-quasihomogeneity can then be reformulated in the following way.

Definition 1.1.6. The system of differential equations (1.3) is called *semi-quasihomogeneous* if, under the action of the flow (1.11), its right hand side is transformed into (1.8), where $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is some quasihomogeneous vector field in the sense of Definition 1.1.5 and where $\mathbf{f}^*(\mathbf{x}, t, \mu)$ is a formal power series with respect to μ^{β}, $\beta \in R \setminus \{0\}$, without a free term. If $\beta > 0$, the system (1.3) will be called *positive semi-quasihomogeneous*; it is called *negative semi-quasihomogeneous* for $\beta < 0$.

We now look at a few simple examples.

Example 1.1.1. The system of ordinary differential equations in the plane

$$\dot{x} = y, \quad \dot{y} = x^2$$

is quasihomogeneous of degree $\mathrm{q} = 2$ with exponents $s_x = 2$, $s_y = 3$. In fact, an arbitrary system of the form

$$\dot{x} = y + f(x, y), \quad \dot{y} = x^2 + g(x, y),$$

where f contains only quadratic or higher terms and g only cubic or higher terms, is semi-quasihomogeneous.

Example 1.1.2. The system of differential equations

$$\dot{x} = (x^2 + y^2)(ax - by), \quad \dot{y} = (x^2 + y^2)(ay + bx)$$

is clearly (quasi-)homogeneous of degree q $= 3$ with exponents $s_x = s_y = 1$. It is, moreover, invariant with respect to the actions of the phase flows of the following family of Fuchsian systems

$$\mu\frac{dx}{d\mu} = \frac{1}{2}x + \delta y, \ \mu\frac{dy}{d\mu} = \frac{1}{2}y - \delta x, \ \mu\frac{dt}{d\mu} = -t.$$

Therefore this system is quasihomogeneous in accordance with Definition 1.1.5 Furthermore, an arbitrary system of the form

$$\dot{x} = (\rho + f(\rho))(ax - by), \quad \dot{y} = (\rho + f(\rho))(ay + bx),$$

where $\rho = x^2 + y^2$ and $f(\rho) = o(\rho)$ as $\rho \to 0$, is semi-quasihomogeneous.

Example 1.1.3. The Emden-Fowler equation [14], which describes the expansion process of a polytropic gas,

$$t\ddot{x} + 2\dot{x} - atx^p = 0, \quad p \geq 2,$$

offers the simplest example of a nonautonomous quasihomogeneous system.

This equation can be rewritten as a system of two equations

$$\dot{x} = t^{-2}y, \quad \dot{y} = at^2x^p,$$

which are quasihomogeneous of degree q $= p$, with exponents $s_x = 2, s_y = 3 - p$.

The need to extend the concepts of quasihomogeneity and semi-quasihomogeneity is dictated by the following circumstance. Situations are frequently encountered where the truncation chosen with the aid of the group of quasihomogeneous coordinate dilations does not have solutions with the desired asymptotic properties. Thus for the choice of "significant" truncations we resort to a procedure whose algorithm is rather fully described in the book by A.D. Bryuno [32]. One of the most important facets of this algorithm is the application to (1.1) of a "birational" transformation that is defined on some cone with singular point, resulting in the possibility of selecting, from the full system, a quasihomogeneous subsystem that has one or another necessary property. The very introduction of the definition extends a priori the best understood truncations. In analyzing planar systems in the neighborhood of a critical point we likewise frequently resort to certain other procedures, specifically to a σ–process or a procedure for blowing up singularities (see the book [7] and also the survey [3]).

We will now examine more closely the quasihomogeneous truncation (1.9) for the original system of differential equations (1.3). Due to quasihomogeneity, in a sufficiently general setting (1.10) has a particular solution in the form of a "quasihomogeneous ray"

$$\mathbf{x}^{\gamma}(t) = (\gamma t)^{-\mathbf{G}} \mathbf{x}_0^{\gamma}, \tag{1.12}$$

where $\gamma = \pm 1$ and $\mathbf{x}_0^{\gamma}$ is a nonzero real vector. Below we will clarify the concept "sufficiently general setting".

If (1.9) has a particular solution of the form (1.12), then the vector $\mathbf{x}_0^{\gamma}$ must satisfy the following algebraic system of equations:

$$-\gamma \mathbf{G} \mathbf{x}_0^{\gamma} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}, \gamma). \tag{1.13}$$

We then say that the vector $\mathbf{x}_0^{\gamma}$ is an *eigenvector* of the quasihomogeneous vector field $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ of the induced quasihomogeneous system $\mathbf{G}$.

The fundamental result of this section amounts to the fact that the system (1.3) has a particular solution that, in a certain sense, is reminiscent of the asymptotic particular solution (1.12) of the truncated system (1.9).

The following assertion generalizes a theorem in [60].

Theorem 1.1.2. *Suppose that system (1.3) is semi-quasihomogeneous and that there exists a nonzero vector $\mathbf{x}_0^{\gamma} \in \mathbb{R}^n$ and a number $\gamma = \pm 1$ such that the equality (1.13) holds. Then the system (1.3) has a particular solution whose principal component has the asymptotic $(\gamma t)^{\mathbf{G}} \mathbf{x}_0^{\gamma}$ as $t^{\chi} \to \gamma \times \infty$, where $\chi = \operatorname{sign} \beta$ is the "semi-quasihomogeneity" index.*

Before proving the above theorem, we discuss its hypothesis. Demonstrating the existence of particular asymptotic solutions for the complete system reduces to the search for eigenvectors of the truncated system. With this interpretation, the hypothesis of the theorem recalls the Lyapunov hypothesis for the existence of exponential solutions obtained from the first approximation system. In our case, this role of the linearized system is played by the quasihomogeneous truncation obtained. The particular solution (1.12) of the truncated system will correspond to the exponential particular solution of the system of first approximation, whose existence in itself implies the existence of a particular solution of the complete system with exponential asymptotic.

In the Lyapunov case the determination of eigenvectors reduces to a well known problem in linear algebra. But in the nonlinear case under investigation the search for such eigenvectors may turn out not to be an easy matter. Nonetheless, their existence in a number of instances follows from rather simple geometric considerations.

From Definition 1.1.5 we have the identity

$$\mathbf{f}_{\mathrm{q}}(\mu^{\mathbf{G}} \mathbf{x}, \mu^{-1} t) = \mu^{\mathbf{G}+\mathbf{E}} \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t). \tag{1.14}$$

Staying with the simple case where the truncated system (1.9) is autonomous: $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t) \equiv \mathbf{f}_{\mathrm{q}}(\mathbf{x})$, where $\mathbf{x} = \mathbf{0}$ is a critical point of this vector field and the matrix $\mathbf{G}$ is nondegenerate. The vector $\mathbf{x}_0^{\gamma}$ will be sought in the following form:

$$\mathbf{x}_0^{\gamma} = \mu^{\mathbf{G}} \mathbf{e}^{\gamma},$$

where μ is a positive number and $\mathbf{e}^{\gamma} \in \mathbb{R}^n$ is a unit vector, i.e. $\|\mathbf{e}^{\gamma}\| = 1$.

Using (1.14), we rewrite Eq. (1.13) in the form

$$\mathbf{G}^{-1}\mathbf{f}_{\mathrm{q}}(\mathbf{e}^{\gamma}) = -\gamma\mu^{-1}\mathbf{e}^{\gamma}. \tag{1.15}$$

Let $\mathbf{x} = \mathbf{0}$ be a critical point of the vector field $\mathbf{f}_{\mathrm{q}}$. We consider the vector field

$$\mathbf{g}(\mathbf{x}) = \mathbf{G}^{-1}\mathbf{f}_{\mathrm{q}}(\mathbf{x}).$$

The following lemma amounts to a "quasihomogeneous version" of assertions from [59, 62, 64, 111].

Lemma 1.1.1. *Let $\mathbf{x} = \mathbf{0}$ be the unique singular point of the autonomous quasihomogeneous vector field $\mathbf{f}_{\mathrm{q}}: \mathbb{R}^n \to \mathbb{R}^n$. We have:*

1. *If the index i of the vector field $\mathbf{g}$ at the point $\mathbf{x} = \mathbf{0}$ is even, then $\mathbf{f}_{\mathrm{q}}$ has eigenvectors with both positive and negative eigenvalues γ,*
2. *If the dimension n of phase space is odd, then $\mathbf{f}_{\mathrm{q}}$ has at least one eigenvector with either a positive or with a negative eigenvalue.*

Proof. To prove the first assertion, we consider the Gauss map

$$\mathbf{\Gamma}(\mathbf{P}) = \frac{\mathbf{g}(\mathbf{P})}{\|\mathbf{g}(\mathbf{P})\|}, \quad \mathbf{P} \in S^{n-1}$$

of the unit sphere S^{n-1} to itself. The index of the vector field $\mathbf{g}$ at the point $\mathbf{x}= 0$ is equal to the degree of the map $\mathbf{\Gamma}$. Consequently the degree of $\mathbf{\Gamma}$ is different from $(-1)^{n-1}$ and therefore $\mathbf{\Gamma}$ has a fixed point [139, 191]. This means that there exists a vector $\mathbf{e}^- \in \mathbb{R}^{\mathbf{n}}$ such that

$$\mathbf{g}(\mathbf{e}^-) = \mu^{-1}\mathbf{e}^-, \quad \mu = \|\mathbf{g}(\mathbf{e}^-)\|^{-1},$$

i.e., (1.15) holds with $\gamma = -1$.

In order to further show that the given map also has the antipodal point ($\gamma = +1$), we need to look at the antipodal map

$$\mathbf{\Gamma}_-(\mathbf{P}) = -\mathbf{\Gamma}(\mathbf{P}),$$

whose degree is also even.

We thus observe that the degree Γ will always be even, provided that each element of $\Im\Gamma$ has an even number of inverse images.

In order to prove the conclusion of item 2, it suffices to examine the tangent vector field

$$\mathbf{v}(\mathbf{P}) = \mathbf{g}(\mathbf{P}) - \langle \mathbf{g}(\mathbf{P}), \mathbf{P}\rangle \mathbf{P}, \quad \mathbf{P} \in S^{n-1}$$

on the sphere S^{n-l} of even dimension (here and in the sequel the symbol $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^n$), on which there does not exist a smooth nonzero vector field [139, 191]. We can thus find a vector $\mathbf{e}^{\gamma} \in \mathbb{R}^n$ such that

$$\mathbf{g}(\mathbf{e}^{\gamma}) = -\gamma\mu^{-1}\mathbf{e}^{\gamma},\ \mu = |\langle \mathbf{g}(\mathbf{e}^{\gamma}), \mathbf{e}^{\gamma}\rangle|^{-1},\ \gamma = -\operatorname{sign}\langle \mathbf{g}(\mathbf{e}^{\gamma}), \mathbf{e}^{\gamma}\rangle.$$

Inasmuch as the vector $\mathbf{g}(\mathbf{e}^{\gamma})$ is parallel to $\mathbf{e}^{\gamma}$ and $\mathbf{x} = \mathbf{0}$ is the unique singular point of the vector field $\mathbf{g}$, we have $\mu^{-1} \neq 0$.

The lemma is thus established.

The index i of a vector field $\mathbf{g}$ will be even if, for instance, $\mathbf{f}_{\mathrm{q}}$ is a homogeneous vector field of even degree with an isolated singular point. Thus a system with a quadratic right hand side *in general* has linear asymptotic solutions. In order to clarify this situation, we consider a simple example.

Example 1.1.4. The conditions for the existence of nontrivial particular solutions of the form

$$x^{\gamma}(t) = (\gamma t)^{-1}x_0^{\gamma}, \quad y^{\gamma}(t) = (\gamma t)^{-1}y_0^{\gamma}$$

for a system of differential equations in the plane

$$\dot{x} = P_1(x, y) = a_1x^2 + b_1xy + c_1y^2,\ \dot{y} = P_2(x, y) = a_2x^2 + b_2xy + c_2y^2$$

may be expressed, for example, in the form of the inequalities:

$$\begin{aligned}\Delta &= a_1^2c_2^2 + c_1^2a_2^2 + a_1c_1b_2^2 + b_1^2a_2c_2 - \\ &\quad -a_1b_1b_2c_2 - b_1c_1a_2b_2 - 2a_1c_1a_2c_2 \neq 0\\ \delta &= (a_1^2 + a_2^2)(c_1^2 + c_2^2) \neq 0.\end{aligned}$$

Since, for fixed y, the resultant of the polynomials $P_1(x, y)$ and $P_2(x, y)$, as functions of x, will equal $y^4\Delta$, we have that the inequality $\Delta \neq 0$ is equivalent to the condition that $P_1(x, y)$ and $P_2(x, y)$, with $y \neq 0$, don't have (even a complex!) common root [187]. If $a_1^2 + a_2^2 \neq 0$, then the polynomials $P_1(x, 0)$ and $P_2(x, 0)$ are simultaneously zero only for $x = 0$. Analogously we consider the case of fixed x. Therefore, by the inequalities established above, the origin $x = y = 0$ will be an isolated critical point of the system considered. It is obvious that, in the six-dimensional parameter space, the dimension of the set for which $\Delta = 0$ or $\delta = 0$ equals five, and thus has measure zero.

Proof of Theorem 1.1.2.

First step. Construction of a formal solution.

We seek a formal solution for the system of differential equations (1.3) in the form of a series

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k(\ln(\gamma t))(\gamma t)^{-k\beta}, \tag{1.16}$$

where the $\mathbf{x}_k$ are polynomial functions of $\ln(\gamma t)$.

We should thus note that the series (1.16) is analogous to series that are used in applying Frobenius's method to the solution of linear systems of differential equations in the neighborhood of a regular singular point [42].

We note the analog of formulas (1.2) and (1.16). The sum in (1.16) is obtained from the corresponding sum in (1.2) with the aid of a logarithmic substitution in time, $t \mapsto \ln(\gamma t)$, i.e. the principles for the construction of solutions of a strictly nonlinear system are analogous to those used in Lyapunov's first method.

We note that if the semi-quasihomogeneity index χ equals $+1$, then the powers of t under the summation sign in (1.16) will be negative; they will be positive in the opposite case.

We show that such a formal particular solution exists. We use the fact that the right hand side of (1.3) can be developed in a formal series in the "quasihomogeneous" forms

$$\mathbf{f}(\mathbf{x}, t) = \sum_{m=0} \mathbf{f}_{q+\chi m}(\mathbf{x}, t).$$

The following identity holds, generalizing (1.14):

$$\mathbf{f}_{q+\chi m}(\mu^{\mathbf{G}}\mathbf{x}, \mu^{-1}t) = \mu^{\mathbf{G}+(1+\beta m)\mathbf{E}}\mathbf{f}_{q+\chi m}(\mathbf{x}, t). \tag{1.17}$$

Using (1.17), we make a substitution of the dependent and independent variables:

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(s), \quad s = (\gamma t)^{-\beta},$$

whereby the original system of equations (1.3) takes the form

$$-\gamma\beta s\mathbf{y}' = \boldsymbol{\gamma}\mathbf{G}\mathbf{y} + \sum_{m=0} s^m \mathbf{f}_{q+\chi m}(\mathbf{y}, \gamma), \tag{1.18}$$

where the prime indicates differentiation with respect to the new independent variable s, and the formal solution (1.16) is converted to

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{x}_k(-1/\beta \ln s)s^k. \tag{1.19}$$

We substitute (1.19) into (1.18) and equate coefficients of s^k. Supposing that the first coefficient $\mathbf{x}_0$ is fixed, for $k = 0$ we obtain

$$-\boldsymbol{\gamma}\mathbf{G}\mathbf{x}_0 = \mathbf{f}_q(\mathbf{x}_0, \gamma).$$

Therefore the existence of the coefficient $\mathbf{x}_0 = \mathbf{x}_0^\gamma$ of the series (1.19) is guaranteed by the conditions of the theorem (see (1.13)). For $k \geq 1$ we have the following system of equations:

$$\frac{d\mathbf{x}_k}{d\tau} - \mathbf{K}_k \mathbf{x}_k = \mathbf{\Phi}_k(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}), \tag{1.20}$$

where the $\mathbf{\Phi}_k$ are certain polynomial vector functions of their arguments, where $\tau = -1/\beta \ln s = \ln(\gamma t)$, $\mathbf{K}_k = k\beta\mathbf{E} + \mathbf{K}$, and where

$$\mathbf{K} = \mathbf{G} + \gamma d_{\mathbf{x}} \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma, \gamma)$$

is the so-called *Kovalevsky matrix* [197].

If it is assumed that all coefficients up to the k–th have been found as certain polynomials in τ, then Φ_k is represented in terms of some known polynomials in τ. The system obtained can be regarded as a system of ordinary differential equations with constant coefficients and polynomial right hand side which, as is known, always has a polynomial particular solution $\mathbf{x}_k(\tau)$, whose degree equals $N_k + S_k$, where N_k is the degree of Φ_k as a polynomial in τ and where S_k is the multiplicity of zero as an eigenvalue of $\mathbf{K}_k$. In this way, the determination of all the coefficients of the series (1.19) can be realized by induction. The formal construction of particular asymptotic solutions for (1.3) is thus complete.

Generally speaking, the coefficients $\mathbf{x}_k(\tau)$ are not determined uniquely, but only within the addition of polynomial functions belonging to the kernel of the differential operator $\frac{d}{d\tau} - \mathbf{K}_k$. Therefore, at each step, we obtain some family of polynomial solutions of (1.20), dependent on S_k arbitrary constants. For this reason, our algorithm generally yields not just one particular solution of (1.3), but a whole manifold of such formal particular solutions.

We note that the expansion (1.16) won't contain powers of the logarithmic "time" γt in two instances: (a) if, among the eigenvalues of the Kovalevsky matrix $\mathbf{K}$, there is no number of the form $-k\beta$, $k \in \mathbb{N}$, and (b) such a number exists, but the projections of the vectors $\mathbf{\Phi}_k$ (which by assumption don't depend on τ) onto the kernel of the operator with matrix $\mathbf{K}_k$ are necessarily zero. The arithmetic properties of the eigenvalues of the Kovalevsky matrix play a role in testing systems of differential equations by the ARS-test. In the literature these eigenvalues are also called *resonances* [1] or *Kovalevsky indices* [197].

We conclude, finally, by looking at some properties of the eigenvalues of the matrix $\mathbf{K}$.

Lemma 1.1.2. *If the truncated system is autonomous, then* -1 *must belong to the spectrum* $\mathbf{K}$.

Proof. Differentiating (1.14) with respect to μ and setting $\mu = 1$, we obtain:

$$d\mathbf{f}_{\mathrm{q}}(\mathbf{x})\mathbf{G}\mathbf{x} = (\mathbf{G} + \mathbf{E})\mathbf{f}_{\mathrm{q}}(\mathbf{x}). \tag{1.21}$$

We denote the vector $\mathbf{f}_{\rm q}(\mathbf{x}_0^\gamma)$ by $\mathbf{p}$. Then, using the identity (1.21) and Eq. (1.13), we find

$$\mathbf{Kp} = \mathbf{Gf}_{\rm q}(\mathbf{x}_0^\gamma) - d\mathbf{f}_{\rm q}(\mathbf{x}_0^\gamma)\mathbf{Gx}_0^\gamma = -\mathbf{f}_{\rm q}(\mathbf{x}_0^\gamma) = -\mathbf{p}.$$

The lemma is thus proved.

This result does not hold in the nonautonomous case.

Example 1.1.5. We return to the Emden-Fowler equation (see Example 1.1.3). The system considered has, for even p, the obvious solution

$$x(t) = x_0 t^{-2\beta}, \quad y(t) = -2\beta x_0 t^{(p-3)\beta},$$
$$\beta = 1/(\rho - 1), \quad x_0 = \left(\tfrac{2(3-p)}{a(p-1)^2}\right)^\beta.$$

The eigenvalues of the Kovalevsky matrix that correspond to this solution are

$$\rho_{1,2} = \frac{\beta}{2}\left(5 - p \pm \sqrt{1 + 16p - 7p^2}\right).$$

It is clear that $\rho_{1,2}$ doesn't reduce to -1 for any choice of p.

Thus, in the autonomous case, the presence of logarithms in the corresponding asymptotic solutions, when (1.3) is positive semi-quasihomogeneous in the sense of Definition 1.1.4, represents the general case, since for $k = {\rm q} - 1$ we have degeneracy of the matrix $\mathbf{K}_k$.

Remark 1.1.1. The properties considered for the eigenvalues of the Kovalevsky matrix clearly don't change if we should likewise consider complex solutions $\mathbf{x}_0^\gamma$ of system (1.13), which accordingly reduces to the series (1.16) with complex coefficients.

The ensuing step in the proof of Theorem 1.1.2 should consist of a proof of convergence of (1.14) or of an asymptotic analysis of its partial sums. These questions are quite profound and so we dedicate a separate section to their discussion.

1.2 Problems of Convergence

In the preceding section we proved, under the hypothesis of Theorem 1.1.2, that Eq. (1.3) has a formal particular solution in the form of series (1.16). If we should succeed in proving convergence of these series on some time interval—or be able to show that they are asymptotic expansions of some solution $\mathbf{x}(t)$ of class $\mathbf{C}^\infty[T, +\infty)$ in the positive semi-quasihomogeneous case, or of class $\mathbf{C}^\infty(0, T^{-1}]$ in the negative semi-quasihomogenous case (with positive γ), where T is a sufficiently large positive number—then Theorem 1.1.2 would be established. Thus we come to the inevitable

Second step. Proof of the existence of a particular solution of system (1.3) with asymptotic expansion (1.16).

The proof of convergence or divergence of the series (1.16) turns out to be a rather difficult task. For instance, the unproved assertions in the previously cited paper of G.V. Kamenkov [93] are concerned with just these convergence questions. We should likewise note that it makes sense to talk about convergence only in the analytic case, where the series (1.4) representing the right side of the system under consideration converges over some complex domain. The standard method of proof in similar situations is by using majorants, which always involves elaborate computations. For the moment we will avoid the question of convergence of (1.16) and prove that these series approximate some smooth solution of the system (1.3) with the required asymptotic properties. The results introduced below originally appeared in the paper [115].

We first put (1.3) into the form (1.18) and change the "time scale":

$$s = \varepsilon\xi, \quad 0 < \varepsilon \ll 1.$$

As a result, this system is rewritten in the form

$$-\gamma\beta\xi\frac{d\mathbf{y}}{d\xi} = \gamma\mathbf{G}\mathbf{y} + \sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{q+\chi m}(\mathbf{y},\gamma). \tag{1.22}$$

If, for the construction of a formal solution of (1.3), it is merely required that the right sides (1.3) be represented as a formal power series (1.4), then we consequently only require that the right sides of (1.18) and the relation (1.22) be functions of class $\mathbf{C}^\infty$, at least in some small neighborhood of the point $s = 0$, $\mathbf{y} = \mathbf{x}_0^\gamma$.

For $\varepsilon = 0$, this system reduces to a truncated system corresponding to (1.9):

$$-\beta\xi\frac{d\mathbf{y}}{d\xi} = \mathbf{G}\mathbf{y} + \gamma\mathbf{f}_q(\mathbf{y},\gamma),$$

which has the particular solution $\mathbf{y}_0(\xi) = \mathbf{x}_0^\gamma$, corresponding to the "quasihomogeneous ray" (1.12).

After the transformation described, the K–th partial sum of the series (1.19) takes the form:

$$\mathbf{y}_K^\varepsilon(\xi) = \sum_{k=0}^{K}\varepsilon^k\mathbf{x}_k\left(-\frac{1}{\beta}\ln(\varepsilon\xi)\right)\xi^k,$$

from which it is obvious that, as $\varepsilon \to +0$, this sum converges to $\mathbf{y}_0(\xi) = \mathbf{x}_0$ uniformly on the interval $[0, 1]$.

Let $K \in \mathbb{N}$ be large enough so that $-\beta K < \Re\rho_i$, $i = 1, \ldots, n$, where the ρ_i are the eigenvalues of the Kovalevsky matrix $\mathbf{K}$.

We will look for a particular solution of (1.22) of the form

$$\mathbf{y}(\xi) = \mathbf{y}_K^\varepsilon(\xi) + \mathbf{z}(\xi)$$

for sufficiently small $\varepsilon > 0$ on the interval [0, 1], with initial condition $\mathbf{y}(+0) = \mathbf{0}$, where $\mathbf{z}(\boldsymbol{\xi})$ has the asymptotic $\mathbf{z}(\xi) = O(\xi^{K+\delta})$ as $\xi \to +0$ and where $\delta > 0$ is fixed but sufficiently small.

We write (1.22) in the form of an equation on a Banach space:

$$\begin{gathered}\Phi(\varepsilon, \mathbf{z}) = \mathbf{0},\\ \Phi(\varepsilon, \mathbf{z}) = \beta\xi \tfrac{d}{d\xi}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \mathbf{G}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \\ +\gamma \textstyle\sum_{m=0} \varepsilon^m \xi^m \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}_K^\varepsilon + \mathbf{z}, \gamma).\end{gathered} \tag{1.23}$$

We regard $\Phi(\varepsilon, z)$ as a mapping

$$\Phi\colon (0, \varepsilon_0) \times \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta},$$

where:

$\mathfrak{B}_{1,\Delta}$ is the Banach space of vector functions $\mathbf{z} : [0, 1] \to \mathbb{R}^n$ that are continuous on [0, 1] along with their first derivatives, and for which the norm

$$\|z\|_{1,\Delta} = \sup_{[0,1]} \xi^{-\Delta}(\|z(\xi)\| + \xi\|z'(\xi)\|)$$

is finite (here the prime indicates differentiation with respect to ξ),

and where

$\mathfrak{B}_{0,\Delta}$ is the Banach space of vector functions $u\colon [0, 1] \to \mathbb{R}^n$ that are continuous on [0, 1] and for which the norm

$$\|u\|_{0,\Delta} = \sup_{[0,1]} \xi^{-\Delta}\|u(\xi)\|,$$

is finite, where $\Delta = K + \delta$.

We note several properties of the map Φ:

(a) $\Phi(0, \mathbf{0}) = \beta\xi \frac{d}{d\xi} y_0(\xi) + G y_0(\xi) + \gamma f_{\mathrm{q}}(y_0(\xi), \gamma) = 0$,
(b) Φ is continuous for ε, $\mathbf{z}$ in $(0, \varepsilon_0) \times \mathfrak{U}_{1,\Delta}$, where $\mathfrak{U}_{1,\Delta}$ is some neighborhood of zero in $\mathfrak{B}_{1,\Delta}$,
(c) Φ is strongly differentiable with respect to $\mathbf{z}$ on$(0, \varepsilon_0) \times \mathfrak{U}_{1,\Delta}$, and its Frechet derivative:

$$\begin{gathered}\nabla_{\mathbf{z}}\Phi(\varepsilon, \mathbf{z})\mathbf{h} = \beta\xi \tfrac{d}{d\xi}\mathbf{h} + \mathbf{Gh} + \gamma \textstyle\sum_{m=0} \varepsilon^m \xi^m d\mathbf{f}_{\mathrm{q}+\boldsymbol{\chi}\mathbf{m}}(\mathbf{y}_{\mathbf{J}}^{\boldsymbol{\varepsilon}} + \mathbf{z}, \boldsymbol{\gamma})\mathbf{h},\\ \mathbf{h} \in \mathfrak{B}_{\mathbf{1},\boldsymbol{\Delta}}\end{gathered}$$

is a bounded operator, continuously dependent on ε, z.

(d) The assertions (a), (b), (c) are rather obvious. The following assertion is less trivial.

Lemma 1.2.1. *The operator* $\nabla_{\mathbf{z}}\Phi(0,\mathbf{0})\colon \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta}$

$$\nabla_{\mathbf{z}}\Phi(0,\mathbf{0}) = \beta\xi\frac{d}{d\xi} + \mathbf{K}$$

has a bounded inverse.

Proof. We prove the existence of a unique particular solution of the system of differential equations

$$\beta\xi\frac{dz}{d\xi} + \mathbf{K}\mathbf{z} = \mathbf{u}, \quad \mathbf{u} \in \mathfrak{B}_{0,\Delta}, \tag{1.24}$$

with initial condition $\mathbf{z}(+0) = 0$, that satisfies the inequality

$$\|\mathbf{z}\|_{1,\Delta} \leq C\|\mathbf{u}\|_{0,\Delta}, \tag{1.25}$$

where the constant $C > 0$ is independent of $\mathbf{u} \in \mathfrak{B}_{0,\Delta}$.

Since the space $\mathbb{R}^n$ decomposes into the direct sum of Jordan subspaces invariant under the linear operator with matrix $\mathbf{K}$, the bound (1.25) suffices for proof in the particular case where $\mathbf{K}$ is a complex Jordan matrix with eigenvalue ρ, $\operatorname{Re}\rho > -\beta\Delta$.

Setting $\widetilde{\rho} = \beta^{-1}\rho$, $\widetilde{\mathbf{u}} = \beta^{-1}\mathbf{u}$, we transform system (1.24) to scalar form:

$$\begin{aligned} &\xi\frac{dz^i}{d\xi} + \widetilde{\rho}z^i + z^{i+1} = \widetilde{u}^i(\xi), \quad i = 1,\dots,n-1 \\ &\xi\frac{dz^n}{d\xi} + \widetilde{\rho}z^n = \widetilde{u}^n(\xi). \end{aligned} \tag{1.26}$$

For the initial conditions $z^1(+0) = \dots = z^n(+0) = 0$, the solution of system (1.26) assumes the following form:

$$\begin{aligned} z^n(\xi) &= \xi^{-\widetilde{\rho}}\int_0^{\xi}\eta^{\widetilde{\rho}-1}\widetilde{u}^n(\eta)d\eta, \\ z^i(\xi) &= \xi^{-\widetilde{\rho}}\int_0^{\xi}\eta^{\widetilde{\rho}-1}(\widetilde{u}^i(\eta) - z^{i+1}(\eta))d\eta, \quad i = 1,\dots,n-1. \end{aligned}$$

The constructed solution, of course, belongs to the space $\mathfrak{B}_{1,\Delta}$. Because $\operatorname{Re}\widetilde{\rho} > -\Delta$, the following bounds hold:

$$\begin{aligned} &\|z^n\|_{1,\Delta} \leq (1 + (1 + |\widetilde{\rho}|))(\Re\widetilde{\rho} + \Delta)^{-1}\|\widetilde{u}^n\|_{0,\Delta}, \\ &\|z^i\|_{1,\Delta} \leq (1 + (1 + |\widetilde{\rho}|))(\Re\widetilde{\rho} + \Delta)^{-1}\|\widetilde{u}^i - z^{i+1}\|_{0,\Delta}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad i = 1,\dots,n-1. \end{aligned}$$

Noting that $\|\cdot\|_{0,\Delta} \leq \|\cdot\|_{1,\Delta}$ and making recursive estimates in each coordinate $z^i(\xi)$, we obtain the inequality (1.25), from which follows the assertion on the existence of a bounded inverse for the operator $\nabla_z\Phi(0,0)$.

The lemma is proved.

Thus all the hypothesis of the abstract theorem on implicit functions [94] is fulfilled so that, for arbitrary $\varepsilon \in (0,\varepsilon_0)$, where $\varepsilon_0 > 0$ is sufficiently small, the

Eq. (1.23) will have a solution in the space $\mathfrak{B}_{1,\Delta}$ that is continuously dependent on ε. Progressing to the variables $\mathbf{y}, s$, we obtain that the differential equation (1.18) has a particular solution $\mathbf{y}(s)$ of class $\mathbf{C}^1[0, \varepsilon]$ with asymptotic

$$\mathbf{y}(s) = \sum_{k=0}^{K} \mathbf{x}_k(-\frac{1}{\beta} \ln s)s^k + o(s^K).$$

In fact, inasmuch as the right side of (1.18) is a smooth vector function, we have that $\mathbf{y} \in \mathbf{C}^\infty[0, \varepsilon]$. Returning to the original variables $\mathbf{x}, t$, we obtain the required smooth existence on $[T, +\infty)$ or on $(0, T^{-1}]$, $T = \varepsilon_0^{-1/\beta}$ (in case $\gamma > 0$), depending on the sign of semi-quasihomogeneity, of a particular solution of the original system with the prescribed principal asymptotic component.

Theorem 1.1.2 is proved.

Remark 1.2.1. The application we have just observed of the implicit function theorem does not permit us to make any claims about the convergence of the series (1.19) nor, consequently, of (1.16).The method of majorant estimates recalled above is perhaps quickest in providing a positive answer to the convergence question for a series given in some neighborhood of $s = 0$. We will examine below a substantially more complicated convergence problem for series constructed in the complex domain.

In our particular situation it can be asserted that the series in question converges, provided that the following requirements are met:

1. The right sides of (1.3) are complex analytic functions on a domain containing the desired solution. Thus the series on the right side of (1.18) represents some holomorphic vector function $\mathbf{y}$ over a neighborhood of $\mathbf{x}_0^\gamma$, for sufficiently small s, $|s| < s_0$.
2. The coefficients of the series (1.19), and consequently those of (1.16), don't depend on the logarithms of the corresponding variables.

This last fact is based on the circumstance that the logarithms in the expansion (1.19) can appear only after the initial steps.

For the proof, it is unavoidable to have to somewhat modify the reasoning introduced above in connection with the implicit function theorem, and precisely to "narrow down" the domain of definition of the map $\mathbf{\Phi}: (0, \varepsilon_0) \times \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta}$, replacing the spaces $\mathfrak{B}_{1,\Delta}$, $\mathfrak{B}_{0,\Delta}$ by the spaces $\mathfrak{C}_{1,K}$, $\mathfrak{C}_{0,K}$, where

$\mathfrak{C}_{1,K}$ is the Banach space of vector functions $\mathbf{z}: \mathcal{K}_1 \to \mathbb{C}^n$, holomorphic on the open unit disk $\mathcal{K}_1 = \{\xi \in \mathbb{C}, |\xi| < 1\}$, continuous on the boundary along with their first derivatives, real on the real axis ($\mathbf{z}(\overline{\boldsymbol{\xi}}) = \overline{\mathbf{z}}(\boldsymbol{\xi})$) and having at the center $\xi = 0$ of the disk a zero of order $K + 1$. In connection with the norm of $\mathfrak{C}_{1,K}$, we consider the expression

$$\|\mathbf{z}\|_{1,K} = \sup_{|\xi|<1} \xi^{-(K+1)}(\|\mathbf{z}(\boldsymbol{\xi})\| + \boldsymbol{\xi}\|\mathbf{z}'(\boldsymbol{\xi})\|),$$

where the prime once again denotes differentiation with respect to ξ.
$\mathfrak{E}_{0,K}$ is the Banach space of vector functions $\mathbf{u}: \mathcal{K}_1 \to \mathbb{C}^n$ holomorphic on the open unit disk $\mathcal{K}_1$, continuous on its boundary, real on the real axis $(\mathbf{u}(\bar{\boldsymbol{\xi}}) = \bar{\mathbf{u}}(\boldsymbol{\xi}))$ and having at the center $\xi = 0$ of the disk a zero of order $K + 1$. In connection with the norm of $\mathfrak{E}_{0,K}$ we consider the expression

$$\|\mathbf{u}\|_{0,K} = \sup_{|\xi| \le 1} \xi^{-(K+1)} \|\mathbf{u}(\boldsymbol{\xi})\|.$$

Subsequent to contracting the domain of definition and the set of values of the mapping $\boldsymbol{\Phi}$, it is likewise possible to apply the implicit function theorem by the strategy already considered. We need only somewhat revise point (d) of the proof.

Lemma 1.2.2. *The Frechet derivative* $\nabla_{\mathbf{z}} \boldsymbol{\Phi}(0, \mathbf{0}): \mathfrak{E}_{1,K} \to \mathfrak{E}_{0,K}$ *has a bounded inverse.*

Proof. We observe that the operator $\nabla_{\mathbf{z}} \Phi(0, \mathbf{0})$ is bounded and we consider a system of differential equations of type (1.24):

$$\beta \xi \frac{d\mathbf{z}}{d\xi} + \mathbf{K}\mathbf{z} = \mathbf{u}, \quad \mathbf{u} \in \mathfrak{E}_{0,K}. \tag{1.27}$$

We expand the function $\mathbf{u}(\xi)$ in a Taylor series

$$\mathbf{u}(\xi) = \sum_{k=K+1}^{\infty} \mathbf{u}_k \xi^k.$$

The solution of (1.27) will likewise be sought in the form of a Taylor series:

$$\mathbf{z}(\xi) = \sum_{k=K+1}^{\infty} \mathbf{z}_k \xi^k.$$

The coefficients of the two series are connected by this relation:

$$\mathbf{u}_k = \mathbf{K}_k \mathbf{z}_k, \quad \mathbf{K}_k = k\beta \mathbf{E} + \mathbf{K}.$$

By virtue of satisfying the inequalities

$$-\beta K < \Re \rho_i, i = 1, \dots, n$$

for arbitrary $k \ge K + 1$, the matrix $\mathbf{K}_k$ is nonsingular, and furthermore has, for large k, the asymptotic estimate

$$\|\mathbf{K}_k\|^{-1} = O(k^{-1}).$$

From this, after application of Cauchy's theorem, it follows that the Taylor series of the functions $\mathbf{z}(\xi)$ and $\mathbf{z}'(\xi)$ have the same radius of convergence as the series for $\mathbf{u}(\xi)$. Therefore the operator $\nabla_{\mathbf{z}}\mathbf{\Phi}(0,\mathbf{0})$ maps the space $\mathfrak{E}_{1,K}$ one-to-one onto the space $\mathfrak{E}_{0,K}$. In consequence of this, by Banach's theorem on the inverse operator [94], $(\nabla_{\mathbf{z}}\mathbf{\Phi}(0,\mathbf{0}))^{-1}$ is bounded.

The lemma is proved.

It is consequently possible to apply the implicit function theorem, which proves the convergence of the series (1.19) as a Taylor series, representing some function holomorphic on $|s| < \varepsilon_0$.

The nonapplicability in general of the procedure described is connected with the fact that the Riemann surface of $\ln t$ is not compact, so that there does not exist a reasonable Banach space of functions that are holomorphic on this surface.

In the general case the logarithm in series (1.16) is "indestructible". However, below we formulate simple conditions that are sufficient for the existence of some "uniformizing" time substitution, subsequent to which the existence of a formal solution can be represented in the form of an ordinary Taylor series. The idea for such a substitution is due to the American mathematician S.D. Taliaferro [182]. The proof of convergence of the series repeats almost exactly the proof introduced above. Therefore the series (1.16), constructed in "real time", also converges.

Theorem 1.2.1. *Suppose that system (1.3) is autonomous, semi-quasihomogeneous in the sense of Definition 1.1.4 and satisfies all the hypothesis of Theorem 1.1.2. Suppose too that the right side terms of system (1.18) are holomorphic on a neighborhood of* $s = 0$, $\mathbf{y} = \mathbf{x}_\mathbf{0}^\gamma$. *If the number* -1 *is the unique solution of the characteristic equation* $\det(\mathbf{K} - \rho\mathbf{E}) = 0$ *of the form* $\rho = -k\beta$, $k \in \mathbb{N}$, *then there exists a particular solution* $\mathbf{x}(t)$*of system (1.3) with asymptotic expansion (1.16) such that* $s^{-\mathbf{S}}\mathbf{x}(t(s))$ *is a vector function holomorphic on the domain* $|s| < \varepsilon_0$, *where* $\varepsilon_0 > 0$ *is sufficiently small,* $\mathbf{S} = (\mathrm{q}-1)\mathbf{G}$, *and* $t(s) = \gamma(s^{1-\mathrm{q}} - \alpha\beta^{-1}\ln s)$, α *being some real parameter.*

In the paper [115], in which this result appears, there are some errors and misprints.

Proof. The function $t(s)$ is the inverse of the solution of the differential equation

$$\dot{s} = -\gamma\beta\frac{s^{\mathrm{q}}}{(1+\alpha s^{\mathrm{q}-1})}, \tag{1.28}$$

satisfying the condition $s(\gamma \times \infty) = 0$.

We make a change of dependent variable $\mathbf{x}(t) = s^{\mathbf{S}}\mathbf{y}(s)$ and a change of independent variable $t \mapsto s$, determined by condition (1.28). Subsequent to this, system (1.3) assumes the form

$$-\gamma\beta s\mathbf{y}' = \gamma\mathbf{G}\mathbf{y} + (1+\alpha s^{\mathrm{q}-1})\sum_{m=0} s^m \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}). \tag{1.29}$$

For the autonomous case with $\alpha = 0$, this system transforms into (1.18). The conformity of systems (1.18) and (1.29) still depends on the solutions of Eq. (1.28) being subject to the condition $s(\gamma \times \infty) = 0$ and having the asymptotic $s(t) \sim (\gamma t)^{-\beta}$.

We will look for a formal particular solution of (1.29) in the form of an ordinary Taylor series

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{y}_k s^k. \tag{1.30}$$

We substitute (1.30) into (1.29) and equate coefficients of like powers of s. For the zero-th power of s we obtain

$$-\mathbf{G}\mathbf{y}_0 = \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0),$$

whereby $\mathbf{y}_0 = \mathbf{x}_0^{\gamma}$.

For the k–th power of s, $k < \mathrm{q} - 1$, we have the equations

$$\mathbf{K}_k \mathbf{y}_k = \mathbf{\Phi}_k(\mathbf{y}_0, \ldots, \mathbf{y}_{k-1}), \quad \mathbf{K}_k = k\beta\mathbf{E} + \mathbf{K}, \tag{1.31}$$

where the quantity Φ_k depends polynomially on its arguments and doesn't depend, for the moment, on the parameter α, which still remains to be determined.

Since for $k \neq \mathrm{q} - 1$ the matrix $\mathbf{K}_k$ is nonsingular, we have that the coefficients $\mathbf{y}_k$ are uniquely determined by the formula

$$\mathbf{y}_k = \mathbf{K}_k^{-1} \mathbf{\Phi}_k(\mathbf{y}_0, \ldots, \mathbf{y}_{k-1}).$$

For $k = \mathrm{q} - 1$ we have:

$$\mathbf{K}_{\mathrm{q}-1}\mathbf{y}_{\mathrm{q}-1} = \alpha \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) + \mathbf{\Phi}_{\mathrm{q}-1}(\mathbf{y}_0, \ldots, \mathbf{y}_{\mathrm{q}-2}), \quad \mathbf{K}_{\mathrm{q}-1} = \mathbf{K} + \mathbf{E}. \tag{1.32}$$

We note that $\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) = \mathbf{p}$, where $\mathbf{p}$, is an eigenvector of the Kovalevsky matrix $\mathbf{K}$ with eigenvalue $\rho = -1$.

We expand $\mathbf{y}_{\mathrm{q}-1}$, $\mathbf{\Phi}_{\mathrm{q}-1}$ into a sum of components, each belonging, respectively, to the eigenspace of the matrix $\mathbf{K}$ generated by the vector $\mathbf{p}$ and to its orthogonal complement:

$$\mathbf{y}_{\mathrm{q}-1} = y_{\mathrm{q}-1}\mathbf{p} + \mathbf{y}_{\mathrm{q}-1}^{\perp}, \ \mathbf{\Phi}_{\mathrm{q}-1} = \phi_{\mathrm{q}-1}\mathbf{p} + \mathbf{\Phi}_{\mathrm{q}-1}^{\perp}.$$

Since the matrix $\mathbf{K}_{\mathrm{q}-1}$ is nonsingular on the invariant subspace orthogonal to the vector $\mathbf{p}$, we have

$$\mathbf{y}_{\mathrm{q}-1}^{\perp} = \mathbf{K}_{\mathrm{q}-1}^{-1} \mathbf{\Phi}_{\mathrm{q}-1}^{\perp}.$$

Setting $\alpha = -\phi_{\mathrm{q}-1}$, we finally satisfy Eq. (1.32). The number $y_{\mathrm{q}-1}$ may now be chosen arbitrarily.

For $k > \mathrm{q} - 1$, the equations for determining $\mathbf{y}_k$ likewise have the form (1.31), where the quantities Φ_k depend on the parameter α determined above. These equations, analogous to those preceding, are easily solved for the $\mathbf{y}_k$ due to the nonsingularity of the matrix $\mathbf{K}_k$.

We have thus shown that Eq. (1.29) has a particular formal solution in the form of a Taylor series (1.30). We now prove that (1.30) is the Taylor series of a function holomorphic on the disk $|s| < \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small.

The proof repeats almost exactly the content of Remark 1.2.1. After the substitution $s = \varepsilon\xi$, $0 < \varepsilon \ll 1$, system (1.29) assumes the form

$$-\gamma\boldsymbol{\beta}\xi\frac{d\mathbf{y}}{d\xi} = \gamma\mathbf{G}\mathbf{y} + (1 + \alpha\varepsilon^{\mathrm{q}-1}\xi^{\mathrm{q}-1})\sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}). \tag{1.33}$$

We let $\mathbf{y}_K^\varepsilon$ denote the partial sum

$$\mathbf{y}_K^\varepsilon(\xi) = \sum_{k=0}^K \varepsilon^k\mathbf{y}_k\xi^k,$$

where $-\boldsymbol{\beta}K < \operatorname{Re}\rho_i$, $i = 1,\dots,n$, and we will seek a particular solution (1.33) in the form

$$\mathbf{y}(\xi) = \mathbf{y}_K^\xi(\xi) + \mathbf{z}(\xi),$$

where $\mathbf{z}(\xi)$ is some function holomorphic on the disk $|\xi| < 1$ having a zero of order $K + 1$ at the point $\xi = 0$.

To prove the existence of such a solution it suffices to apply the implicit function theorem [94] to the Banach space equation

$$\Phi(\varepsilon,\mathbf{z}) = \mathbf{0},$$
$$\Phi\colon (0,\varepsilon_0)\times\mathfrak{E}_{1,k} \to \mathfrak{E}_{0,k},$$

where

$$\Phi(\varepsilon,\mathbf{z}) = \gamma\beta\xi\tfrac{d}{d\xi}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \mathbf{G}(\mathbf{y}_K^\varepsilon + \mathbf{z}) +$$
$$+\gamma(1 + \alpha\varepsilon^{\mathrm{q}-1}\xi^{\mathrm{q}-1})\sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}_K^\varepsilon + \mathbf{z}).$$

Equation (1.28) has a particular solution $s(t)$, $s(+\infty) = 0$, given by the function inverse to

$$t(s) = \gamma(s^{1-\mathrm{q}} - \alpha\boldsymbol{\beta}^{-1}\ln s).$$

Let $\mathcal{R}$ be the Riemann surface of the function $s(t)$. The system of equations (1.29) has a holomorphic particular solution $\mathbf{y}(s)$, so that the vector function $\mathbf{y}(s(t))$ is holomorphic on the encompassing region $|t| > \varepsilon_0^{-1/\beta}$ of the Riemann surface $\mathcal{R}$, on which (1.19) provides an asymptotic solution.

The theorem is proved.

Thus in this section we have completed the proof of Theorem 1.1.2, which has important applications to the theory of the stability of motion. We have the following assertion:

Theorem 1.2.2. *Let* $\mathbf{x} = \mathbf{0}$ *be a critical point of the system (1.3) and let the system (1.3) be autonomous. Suppose too that the system (1.3) is positive* *semi-quasihomogeneous with respect to the quasihomogeneous structure given by the matrix* $\mathbf{G}$*, whose eigenvalues have positive real parts. If there exists a vector* $\mathbf{x}_0^- \in \mathbb{R}^n$, $\mathbf{x}_0^- \neq \mathbf{0}$ *such that*

$$\mathbf{G}\mathbf{x}_0^- = \mathbf{f}_\mathrm{q}(\mathbf{x}_0^-)$$

(i.e. $\gamma = -1$*), then the critical point* $\mathbf{x} = \mathbf{0}$ *is unstable.*

The proof follows immediately from the existence of a particular solution $t \mapsto x(t)$ such that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$.

At the present time, this is the most general result we have connecting stable equilibrium in the total system to stable equilibrium in the truncated (so-called *model*) system. Here we actually prove that the existence of a particular solution of the type of an increasing quasihomogeneous ray for the model system implies, by itself, the unstable equilibrium of the total system. Until recently this assertion was merely a hypothesis whose proof was attempted by many authors, although basically only for "semihomogeneous" systems. In this connection it is worth mentioning the results of Kamenkov [93] already cited, whose proofs contain a number of lacunae, as was mentioned. In the book [100] there is given a rather simple proof of this assertion for the case of an attracting ray. The more general case where the ray is hyperbolic is much more complicated. The corresponding result is proved in the article [170] with the help of sophisticated topological techniques. (The terms "attracting ray" and "hyperbolic ray" are connected with the distribution of the eigenvalues of the Kovalevsky matrix over the complex plane.) Theorem 1.2.2 proves the indicated hypothesis completely.

In his classical study [133], A.M. Lyapunov considered the more general problem of the stability of solutions with respect to a given function of the state of the system. We will show how the approach we have developed can be applied in this more general situation [190]. To this end we examine the smooth system (1.1) with critical point $\mathbf{x} = \mathbf{0}$, so that $\mathbf{x} = \mathbf{0}$ is an equilibrium position. We then have its stability with respect to a smooth (i.e. infinitely differentiable) function $Q(\mathbf{x})$, where it is assumed that $Q(\mathbf{0}) = 0$.

We introduce a new system of equations

$$\dot{\mathbf{x}} = -\mathbf{f}(\mathbf{x}), \tag{1.34}$$

obtained from (1.1) by time reversal.

Lemma 1.2.3. *Suppose that the system (1.34) admits a solution* $t \mapsto \mathbf{x}(t)$ *such that 1)*$\mathbf{x}(t) \to \mathbf{0}$ *as* $t \to +\infty$ *and, for all* t*, 2)* $q(t) = Q(\mathbf{x}(t)) \neq 0$*. Then the equilibrium point* $\mathbf{x} = \mathbf{0}$ *of the system (1.1) is stable with respect to the function* Q*.*

In fact, in this case there is a solution $t \mapsto \mathbf{x}(-t)$, which asymptotically "exits" from the equilibrium state: $q(t) \to 0$ as $t \to -\infty$. Consequently, along this solution the function Q changes continuously from zero to appreciable finite values. But this then signifies instability with respect to Q.

An asymptotic solution of Eq. (1.34) can be sought in the form of a series of a certain form. Let **A** be the Jacobian matrix of some vector field **f** at zero. If this matrix has a positive real eigenvalue β, then system (1.34) admits an asymptotic solution in the form of the series (1.2). Substituting this series into the Maclaurin series of the function Q, we again obtain a series in powers of $\exp(-\beta t)$, whose coefficients are polynomials in t. If at least one of the coefficients of this series is distinct from zero, then (by Lemma 1.2.3) the equilibrium of system (1.1) is unstable with respect to the function Q.

This observation can be generalized. The necessary condition for the stability of the equilibrium point $\mathbf{x} = \mathbf{0}$ for system (1.1) with respect to the function Q is implied by the constancy of this function on the unstable manifold of system (1.1). The last property is verified constructively by means of an iterative method for constructing Lyapunov series that represent asymptotic solutions (as $t \to -\infty$) of system (1.1).

In degenerate cases, asymptotic solutions of system (1.34) can be sought in the form of the series (1.16). The conditions for the existence of such solutions is given by Theorem 1.2.1. The next assertion generalizes Theorem 1.2.2, but prior to this we give a generalized definition of quasihomogeneous and semi-quasihomogeneous functions.

Let **G** be some real matrix from (1.10) yielding a quasihomogeneous structure on $\mathbb{R}^n[\mathbf{x}]$.

Definition 1.2.1. The function $Q : \mathbb{R}^n \to \mathbb{R}$ is called *quasihomogeneous* of degree m, provided that

$$Q(\mu^{\mathbf{G}}\mathbf{x}) = \mu^m Q(\mathbf{x}) \tag{1.35}$$

for all $\mu > 0$.

If $G = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, then the relation (1.35) takes the following explicit form:

$$Q(\mu^{\alpha_1} x_1, \ldots, \mu^{\alpha_n} x_n) = \mu^m Q(x_1, \ldots, x_n).$$

For $\alpha_1 = \ldots = \alpha_n = 1$ we have an ordinary homogeneous function of degree m.

Differentiating (1.35) with respect to μ and setting $\mu = 1$, we get a generalized *Euler identity*

$$\left\langle \frac{\partial Q}{\partial \mathbf{x}}, \mathbf{G} \right\rangle = mQ.$$

Definition 1.2.2. The function Q is said to be positive (negative) semi-quasihomogeneous if it can be represented in the form $Q_m(\mathbf{x}) + \tilde{Q}(\mathbf{x})$, where Q_m is a quasihomogeneous function of degree m and

$$\mu^{-m}\tilde{Q}(\mu^{\mathbf{G}}\mathbf{x}) \to 0$$

as $\mu \to 0$ ($\mu \to +\infty$).

Theorem 1.2.3. *Let all the conditions for Theorem 1.2.2 be satisfied for system (1.34) and let Q be a smooth positive semi-quasihomogeneous function, where*

$$Q_m(\mathbf{x}_0) \neq 0.$$

Then the equilibrium point $x = 0$ of the original system (1.1) is unstable with respect to the function Q.

This assertion is a simple consequence of Theorem 1.1.2 and Lemma 1.2.3.

We now suppose that system (1.34) admits an asymptotic solution in the form of the series (1.16). Then the substitution of this series into the Maclaurin series of the infinitely differentiable function Q gives us an expansion of the function $t \mapsto q(t)$ as a series with a convenient form (with reciprocal degrees t^{β} and coefficients that are polynomials in "logarithmic time" with constant coefficients). If at least one coefficient of this formal series is different from zero, then the trivial equilibrium point of system (1.1) is unstable with respect to the function Q. We will return to these issues in Chap. 3.

1.3 Exponential Methods for Finding Nonexponential Solutions

In the preceding two sections we have explained how to construct particular solutions of differential equations, whose principal asymptotic parts were determined by the quasihomogeneous structure of the chosen truncation. The algorithms introduced allow effective construction of solutions in the form of series. However, it is obvious even at first glance that the given algorithm is far from giving all solutions with the required asymptotic. As series of type (1.2) don't exhaust all solutions of exponential type, so also the series (1.16) don't describe all solutions with a generalized power asymptotic. In the quasilinear case, all exponential solutions lie on the stable and unstable manifolds $W^{(s)}$, $W^{(u)}$. For observing the "strong nonlinearity", we establish results related to the Hadamard-Perron theorem, i.e. we attempt to apply techniques that are typical in searching for solutions with exponential asymptotic.

In this section we alter somewhat the conditions that are imposed on the right hand side of (1.3). Let (1.3) either be autonomous or suppose that the right hand side is a bounded function of t over the entire real line.

We begin with the autonomous case. We have

Theorem 1.3.1. *Suppose that the right hand side of (1.3) doesn't explicitly depend on t. Suppose that all the hypothesis of Theorem 1.1.2 is fulfilled and that l eigenvalues of the Kovalevsky matrix $\mathbf{K}$ have real parts whose sign agrees with the sign of the quantity $-\beta$, at the same time as the real parts of the remaining* *eigenvalues are either zero or have the opposite sign. Then (1.3) has an l–parameter family of particular solutions of the form*

$$\mathbf{x}(\mathbf{c}, t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^\gamma + o(1)) \ \ as \ \ t^\chi \to \gamma \times \infty,$$

where $\mathbf{c} \in \mathbb{R}^l$ *is a vector of parameters.*

This result is a consequence of the more general Theorem 1.3.3 that we formulate and prove below.

There is still another question connected with the search for asymptotic solutions using the algorithms described above. In Sect. 1.1 we sought particular solutions of the truncated quasihomogeneous system (1.9) in the form (1.12). However, it is unknown whether these exhaust all possible solutions of a quasihomogeneous system having the generalized power asymptotic (1.12). It might happen that none of the possible quasihomogeneous truncations has a particular solution in the form of a quasihomogeneous ray, while the total system has an asymptotic solution. This leads us to consider the question of existence of particular solutions of truncated type (1.9) in "nonstationary" form, more general than (1.12), and precisely

$$\mathbf{x}^\gamma(t) = (\gamma t)^{-\mathbf{G}}\mathbf{x}_0^\gamma(\gamma t), \tag{1.36}$$

where $\mathbf{x}_0^\gamma(\cdot)$ is an infinitely differentiable vector function bounded on the positive half-line. We show, with some rather weak restrictions, that these generate solutions of the total system with analogous asymptotic. We likewise extend the class of systems studied by considering nonautonomous systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1.37}$$

where the components of the right sides admit expansion in formal power series:

$$f^j = \sum_{i_1,\dots,i_n} f^j_{i_1,\dots,i_n}(t)(x^1)^{i_1}\dots(x^n)^{i_n}, \tag{1.38}$$

with coefficients $f^j_{i_1,\dots,i_n}(t)$ that are smooth $\mathbb{R}[t]$ functions that are uniformly bounded over the entire axis.

Next, let system (1.37), *where the time t on the right sides is regarded as a parameter*, be semi-quasihomogeneous in the sense of Definition 1.1.6 with respect to a quasihomogeneous structure induced by some matrix $\mathbf{G}$. We will consider the truncated system

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t) \tag{1.39}$$

and look for its particular solutions in the form (1.36).

Making the logarithmic substitution $\tau = \ln(\gamma t)$ for the independent variable, we find that in the "new time" the vector function $\mathbf{x}_0^\gamma(\cdot)$ is a particular solution of the system of differential equations

$$\frac{d\mathbf{x}_0^\gamma}{d\tau} = \mathbf{G}\mathbf{x}_0^\gamma + \gamma\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma, \gamma e^\tau). \tag{1.40}$$

Linearizing (1.40) in the neighborhood of some solution $\mathbf{x}_0^\gamma(\cdot)$, we obtain the linear system

$$\frac{d\mathbf{u}}{d\tau} = \mathbf{K}(\tau)\mathbf{u}, \tag{1.41}$$

where

$$\mathbf{K}(\tau) = \mathbf{G} + \gamma d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma(\gamma e^\tau), \gamma e^\tau),$$

with nonautonomous Kovalevsky matrix $\mathbf{K}(\tau)$, whose components—in view of the boundedness of $\mathbf{x}_0^\gamma$ and of the coefficients in expansion (1.38)—are smooth and bounded over all values of τ.

We have the following assertion, which is analogous to Lemma 1.1.2.

Lemma 1.3.1. *If the system (1.39) is autonomous, then the system (1.41) has the particular solution*

$$\mathbf{u}_0(\tau) = e^{-\tau}\mathbf{p}(\tau), \qquad \mathbf{p}(\tau) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma(\gamma e^\tau)).$$

Proof. Indeed, from the definition of the nonautonomous Kovalevsky matrix, using system (1.40) and Eq. (1.21), we obtain

$$\begin{aligned}\tfrac{d\mathbf{u}_0}{d\tau} &= e^\tau\left(-\mathbf{p} + d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma)\tfrac{d\mathbf{x}_0^\gamma}{d\tau}\right) = \\ &= e^{-\tau}(-\mathbf{p} + d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma)(\mathbf{G}\mathbf{x}_0^\gamma + \gamma\mathbf{p})) = \\ &= e^{-\tau}(\mathbf{G} + \gamma d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma))\mathbf{p} = \mathbf{K}\mathbf{u}_0.\end{aligned}$$

The lemma is proved.

If the bounded vector function $p(\tau)$ doesn't tend to zero as $\tau \to \to \pm\infty$, then the characteristic exponent of the solution $\mathbf{u}_0(\tau)$ of (1.41) equals -1, so that generally -1 belongs to the full spectrum of the linear system (1.41)

We recall briefly several concepts from the theory of the asymptotic behavior of solutions of nonautonomous linear systems of type (1.41). For closer acquaintance with this subject, we recommend the corresponding sections of the book [44].

The *right characteristic exponents* of a scalar or vector valued function $\mathbf{u}(\tau)$ of arbitrary dimension are given by the quantities

$$r^+ = \kappa^+[\mathbf{u}(\tau)] = \limsup_{\tau\to+\infty} \tau^{-1}\ln\|\mathbf{u}(\tau)\|,$$

the corresponding *left characteristic exponents* being given by

$$r^- = \kappa^-[\mathbf{u}(\tau)] = \kappa^+[\mathbf{u}(-\tau)].$$

In what follows, unless otherwise stipulated, we will be dealing with "right" exponents, which determine the asymptotic behavior of a function as $\tau \to +\infty$, whereby the transition to "left" (the case $\tau \to -\infty$) is effected by the change of independent variable $\tau \mapsto -\tau$.

We note some properties of the quantities that have been introduced.

For any vector function $\mathbf{u}(\tau)$ and any constant τ_0,

$$\kappa[\mathbf{u}(\tau - \tau_0)] = \kappa[\mathbf{u}(\tau)].$$

Let $\mathbf{u}_{(1)}(\tau), \mathbf{u}_{(2)}(\tau)$ be two vector functions with finite characteristic exponents. The characteristic exponents of their linear combinations $c_1\mathbf{u}_{(1)}(\tau) + c_2\mathbf{u}_{(2)}(\tau)$ and of their general "composition" $\mathbf{B}(\mathbf{u}_{(1)}(\tau), \mathbf{u}_{(2)}(\tau))$, where $\mathbf{B} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is some bilinear vector function, possess the following properties:

$$\kappa[c_1\mathbf{u}_{(1)}(\tau) + c_2\mathbf{u}_{(2)}(\tau)] = \max\left(\kappa[\mathbf{u}_{(1)}(\tau)], \kappa[\mathbf{u}_{(2)}(\tau)]\right)$$
$$\kappa[\mathbf{B}(\mathbf{u}_{(1)}(\tau), \mathbf{u}_{(2)}(\tau))] \le \kappa[\mathbf{u}_{(1)}(\tau)] + \kappa[\mathbf{u}_{(2)}(\tau)].$$

If $\mathbf{u}(\tau)$ is a vector function with finite nonnegative characteristic exponents, then

$$\kappa\left[\int_0^\tau \mathbf{u}(\xi)d\xi\right] \le \kappa[\mathbf{u}(\tau)];$$

but if a characteristic exponent of $\mathbf{u}(\tau)$ is negative, then

$$\kappa\left[\int_\tau^{+\infty} \mathbf{u}(\xi)d\xi\right] \le \kappa[\mathbf{u}(\tau)].$$

Furthermore, let $\mathbf{U}(\tau) = (u^i_j(\tau))^n_{i,j=1}$ be some matrix with smooth bounded components. Its characteristic exponents are given by

$$\kappa\left[\mathbf{U}(\tau)\right] = \max_{i,j}\left(\kappa\left[u^j_i(\tau)\right]\right) = \max_{i,j}\left(\limsup_{\tau\to+\infty}\tau^{-1}\ln|u^i_j(\tau)|\right).$$

The *full spectrum* of the linear system (1.41) is defined to be the set of quantities $\left\{r_i = \kappa\left[\mathbf{u}_{(i)}(\tau)\right]\right\}^n_{i=1}$, where $\left\{\mathbf{u}_{(i)}(\tau)\right\}^n_{i=1}$ is some fundamental system of its solutions. If system (1.41) is autonomous, then $r_i = \mathrm{Re}\,\rho_i$, where the ρ_i are the roots of the characteristic equation $\det(\mathbf{K} - \rho\mathbf{E}) = 0$.

It is clear that the full spectrum of the system (1.41) depends on the choice of a fundamental system of solutions. For any fundamental system of solutions, the so-called Lyapunov inequality [133] applies:

$$\sum_{i=1}^n r_i \ge \limsup_{\tau\to+\infty}\tau^{-1}\int_0^\tau \mathrm{Tr}\,K(\xi)d\xi, \tag{1.42}$$

where $\mathrm{Tr}\,K$ denotes the trace of the matrix $\mathbf{K}$.

There exist fundamental systems of solutions, called *normal*, for which the sum of the characteristic exponents is maximal [44].

For any normal system of solutions $\{\mathbf{u}_i(\tau)\}_{i=1}^n$, the *irregularity measure* is defined as the quantity

$$\sigma = \sum_{i=1}^{n} r_i - \liminf_{\tau \to +\infty} \tau^{-1} \int_0^{\tau} \operatorname{Tr} K(\xi) d\xi,$$

which is nonnegative by virtue of the Lyapunov inequality (1.42).

The system (1.41) is called *proper* if, for some normal system of its solutions, the irregularity measure σ equals zero, in which case strict equality holds in Lyapunov's inequality (1.42).

The definitions and concepts introduced above are concerned with the asymptotic behavior of solutions of the system (1.41) as $\tau \to +\infty$. In order to introduce analogous characteristics for the case $\tau \to -\infty$, we need to consider the system with time reversal:

$$\frac{d\mathbf{u}}{d\tau} = -\mathbf{K}(-\tau)\mathbf{u}. \tag{1.43}$$

It will be shown below that the task of looking for particular solutions of the system (1.37) that have nonexponential asymptotic can be reduced to the investigation of some quasilinear system, whose linear part has the form (1.41), i.e. by the application of "exponential" methods. We will formulate some results that generalize known theorems of V.I. Zubov [203].

We first prove an analog of Theorem 1.1.2 for system (1.37).

Theorem 1.3.2. *Let the quasihomogeneous truncation (1.39) of the semi-quasihomogeneous system (1.37) have a particular solution of form (1.36) and let the irregularity measure of system (1.41) for the case* $\beta > 0$ *(or the irregularity measure of system (1.43) for the case* $\beta < 0$*) satisfy the inequality*

$$\sigma < \frac{|\beta|}{2}. \tag{1.44}$$

Then (1.37) has a particular solution whose principal part has asymptotic

$$(\gamma t)^{-\mathbf{G}} \mathbf{x}_0^{\gamma}(\gamma t)$$

as $t^{\chi} \to \gamma \times \infty$.

The proof is divided into two parts, just as in the preceding section.

First step. Construction of a formal solution.

We first construct a formal solution for system (1.37) in the form of the series

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta}, \tag{1.45}$$

where $\mathbf{x}_k(\tau)$, $\tau = \ln(\gamma t)$ are some vector valued functions on the entire number line which satisfy the inequality

$$\chi\kappa^{\chi}\left[\mathbf{x}_k(\tau)\right] \leq (2k-1)\sigma. \tag{1.46}$$

The symbol κ^{χ} here denotes the characteristic exponent κ^+ or κ^- in accordance with the sign of the semi-quasihomogeneity.

Since inequality (1.44) is satisfied, the inequality (1.46) guarantees that the desired solution, at least formally, has the required asymptotic.

We change the dependent and independent variables $\mathbf{x} \mapsto \mathbf{u}$, $t \mapsto \tau$,

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^{\gamma}(\gamma t) + \mathbf{u}(\gamma)),\ \ \tau = \ln(\gamma t),$$

subsequent to which the system (1.37) is rewritten

$$\mathbf{u}' = \mathbf{K}(\tau)\mathbf{u} + \phi(\mathbf{u},\tau) + \psi(\mathbf{u},\tau), \tag{1.47}$$

where the prime indicates differentiation with respect to the new "time" τ, where

$$\phi(\mathbf{u},\tau) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma} + \mathbf{u}, \gamma e^{\tau}) - \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}, \gamma e^{\tau}) - d_{\mathrm{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}, \gamma e^{\tau})\mathbf{u}.$$

It is clear that $\phi(\mathbf{u},\tau)$ is a bounded vector function τ for all fixed finite $\mathbf{u}$ and that, moreover, $\phi(\mathbf{u},\tau) = O(\|\mathbf{u}\|^2)$ as $\mathbf{u} \to \mathbf{0}$ uniformly in τ.

The vector function $\psi(\mathbf{u},\tau)$ admits a formal expansion in the series

$$\psi(\mathbf{u},\tau) = \sum_{m-1} e^{-m\beta\tau}\mathbf{f}_{\mathrm{q}+m\chi}(\mathbf{x}_0^{\gamma} + \mathbf{u}, \gamma e^{\tau}).$$

As before, we assume that the right sides of (1.37) are smooth, at least over some domain containing the desired solution. Therefore $\psi(\mathbf{u},\tau) = O(e^{-\beta\tau})$ as $\tau \to \chi \times \infty$ for each fixed $\mathbf{u}$.

We have the following

Lemma 1.3.2. *Under the assumptions that were made above concerning properties of the matrix $\mathbf{K}(\tau)$ and the vector functions $\phi(\mathbf{u},\tau)$ and $\psi(\mathbf{u},\tau)$, the system of equations (1.47) has a formal particular solution of the form*

$$\mathbf{u}(\tau) = \sum_{k=1}^{\infty} \mathbf{x}_k(\tau)e^{-k\beta\tau}, \tag{1.48}$$

where the characteristic exponents of the coefficients satisfy the inequality (1.46).

We restrict ourselves to examining the case $\chi = +1$; the opposite case is reduced to this one by means of the "logarithmic time" transformation: $\tau \mapsto -\tau$.

We substitute (1.48) into (1.47) and equate coefficients of corresponding powers $e^{-\beta\tau}$. The coefficients $\mathbf{x}_k(\tau)$ of the series (1.48) are found inductively. We assume that $\mathbf{x}_1(\tau), \dots, \mathbf{x}_{k-1}(\tau)$ have been found and that their characteristic exponents satisfy inequality (1.46) We write the differential equation for determining $\mathbf{x}_k(\tau)$:

$$\mathbf{x}_k' - \mathbf{K}(\tau)\mathbf{x}_k = \Phi_k(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \tau), \tag{1.49}$$

where $\mathbf{K}_k(\tau) = k\beta\mathbf{E} + \mathbf{K}(\boldsymbol{\tau})$ and the Φ_k are certain vector functions that are bounded in τ and are polynomially dependent on $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$.

Using the properties of the characteristic exponents of a linear combination and a generalized product, we can prove that the characteristic exponents of the vector functions Φ_k, after the substitution into them of $\mathbf{x}_1(\tau), \dots, \mathbf{x}_{k-1}(\tau)$, satisfy the inequalities

$$\kappa\,[\Phi_k(\tau)] = \kappa\,[\Phi_k(\mathbf{x}_1(\tau), \dots, \mathbf{x}_{k-1}(\tau), \tau)] \leq (2k-2)\sigma. \tag{1.50}$$

The index "$+$" will henceforth be dropped in order to shorten the notation. We prove that the system of differential equations (1.49) has a particular solution for which the characteristic exponents satisfy inequality (1.46).

Let $\mathbf{U}(\tau)$ be the fundamental matrix of the system (1.41), normalized by the condition $\mathbf{U}(0) = \mathbf{E}$. Let $r_1, \dots, r_n$ be the full spectrum of the system (1.41). We consider the diagonal matrix $\mathbf{R} = \operatorname{diag}(r_1, \dots, r_n)$.

We perform the substitutions

$$\mathbf{x}_k = e^{k\beta\tau}\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{y}_k,$$
$$\Psi_k(\tau) = e^{-k\beta\tau}\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)\Phi_k(\tau).$$

In the new variables Eq. (1.49) takes on the form

$$\mathbf{y}_k' - \mathbf{R}\mathbf{y}_k = \Psi_k(\tau). \tag{1.51}$$

In conformity with the bound (1.50) and the general properties of characteristic exponents,

$$\kappa[\Psi_\tau(\tau)] \leq -k\beta + (2k-2)\sigma + \kappa[\exp(\mathbf{R}_\tau)\mathbf{U}^{-1}(\tau)].$$

Using the formula for the inverse matrix, we have

$$\mathbf{U}^{-1}(\tau) = (\Delta^{-1}(\tau)\Delta_i^j(\tau))_{i,j=1}^n,$$

where $\Delta(\tau) = \det\mathbf{U}(\tau)$, and the $\Delta_j^i(\tau)$ are the cofactors of the elements $u_j^i(\tau)$.

Using $\Delta(0) = 1$ and the Ostrogradski-Liouville formula, we will have

$$\Delta(\tau) = e^{\int_0^\tau \operatorname{Tr}\mathbf{K}(\boldsymbol{\xi})\mathbf{d}\boldsymbol{\xi}}.$$

We recall that the columns of the fundamental matrix $\mathbf{U}(\tau)$ are vector functions belonging to a fundamental system of solutions $\left\{\mathbf{u}_{(i)}(\tau)\right\}_{i=1}^{n}$, so that

$$\kappa\left[\exp\left(\mathbf{R}\tau\right)\mathbf{U}^{-1}\left(\tau\right)\right] = \max_{i,j}\left(\kappa\left[e^{r_i\tau}\Delta_i^j\left(\tau\right)e^{-\int_0^\tau \mathrm{Tr}\,\mathbf{K}(\boldsymbol{\xi})\mathbf{d}\boldsymbol{\xi}}\right]\right) \leq$$
$$\leq \max_{i,j}\left(r_i + \sum_{l=1}^{n} r_l - r_i - \underline{\lim}_{\tau\to+\infty}\tau^{-1}\int_0^\tau \mathrm{Tr}\,\mathbf{K}(\boldsymbol{\xi})\mathbf{d}\boldsymbol{\xi}\right) = \sigma.$$

Consequently,

$$\kappa[\Psi_k(\tau)] \leq -k\beta + (2k-1)\sigma.$$

We rewrite (1.51) in coordinate form

$$(y_k^i)' - r_i y_k^i = \Psi_k^i(\tau),\ i = 1,\ldots,n,$$

and construct a particular solution of this system by the following recipe:

$$y_k^i(\tau) = -e^{r_i\tau}\int_\tau^{+\infty} e^{-r_i\xi}\Psi_k^i(\xi)d\xi$$

for those i for which $\kappa\left[e^{-r_i\tau}\Psi_k^i(\tau)\right] < 0$, and

$$y_k^i(\tau) = e^{r_i\tau}\left(c_k^i + \int_0^\tau e^{-r_i\xi}\Psi_k^i(\xi)d\xi\right)$$

for those i for which $\kappa\left[e^{-r_i\tau}\Psi_k^i(\tau)\right] \geq 0$, where $c_k^i \in R$ are free parameters that can be chosen arbitrarily if the characteristic exponent appearing in the integral formula is nonnegative, and that we set equal to zero in the opposite case.

From the given formulas it is clear that the characteristic exponent of the solution of (1.51), constructed by the given method, satisfies the bound

$$\kappa[\mathbf{y}_k(\tau)] \leq -k\beta + (2k-1)\sigma.$$

We next compute the characteristic exponents of $\mathbf{U}(\tau)\exp\left(-\mathbf{R}\tau\right)$:

$$\kappa[\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)] = \max_{i,j}\left(\kappa\left[u_j^i(\tau)e^{-r_i\tau}\right]\right) = 0.$$

Therefore $\mathbf{x}_k = e^{k\beta\tau}\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{y}_k$ has a characteristic exponent that satisfies the inequality (1.46).

Thus $\mathbf{x}_k(\tau)$ is the desired solution of (1.49) and the construction of the series (1.48) has been accomplished.

Lemma 1.3.2 is proved.

We have, moreover, proved that the original system (1.37) has a formal solution of the form (1.45).

Second step. We now pass to the construction of the actual solution.

Lemma 1.3.3. *Let the parameter β in the right side of (1.47) be positive. If the conditions of Lemma 1.3.2 are satisfied, then the system of equations (1.47) has a particular solution of the form*

$$\mathbf{u}(\tau) = \mathbf{u}_K(\tau) + \mathbf{v}(\tau),$$

where $\mathbf{u}_K(\tau)$ is the K-th partial sum of the series (1.48) and $\mathbf{v}(\tau)$ has asymptotic

$$\mathbf{v}(\tau) = o\left(e^{-K\beta\tau}\right) \text{ as } \tau \to +\infty,$$

where $b = \beta - 2\sigma$ and K is sufficiently large.

Proof. We write a system of differential equations for $\mathbf{v}$:

$$\mathbf{v}' = \mathbf{K}(\tau)\mathbf{v} + \theta(\mathbf{v}, \tau). \tag{1.52}$$

and introduce the notation

$$\theta(\mathbf{v}, \tau) = -\mathbf{u}_K'(\tau) + \mathbf{K}(\tau)\mathbf{u}_K(\tau) + \phi(\mathbf{u}_K(\tau) + \mathbf{v}, \tau) + \psi(\mathbf{u}_k(\tau) + \mathbf{v}, \tau),$$

where K is chosen so that the inequality $Kb > -r_i$ is satisfied for $i = 1, \ldots, n$.

Using the substitution

$$\mathbf{v} = \mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{w},$$

we dispose of the nonautonominity in the linear portion of (1.52), after which (1.52) takes on the form

$$\mathbf{w}' = \mathbf{R}\mathbf{w} + \widehat{\theta}(\mathbf{w}, \tau), \tag{1.53}$$

where

$$\hat{\theta}(\mathbf{w}, \tau) = \exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)\theta(\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{w}, \tau).$$

We will show that the system of equations (1.53) has a particular solution $\mathbf{w}(\tau)$ that is determined on some half line $[T, +\infty)$, where $T > 0$ is sufficiently large, with asymptotic $\mathbf{w}(\tau) = O(e^{-\Delta\tau})$ as $\tau \to +\infty$, where $\Delta = Kb + \delta$ and $\delta > 0$ is sufficiently small.

Under the assumptions made, we can assert that $\widehat{\theta}(\mathbf{w}(\tau), \tau)$ generally has a higher order of decay than $\mathbf{w}(\tau)$. Because the characteristic exponent of the matrix $\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)$ equals zero, we can write $\mathbf{v}(\tau) = O(e^{-(Kb+\frac{\delta}{2})\tau})$.

We will estimate $\theta(\mathbf{v}, \tau)$ using the mean value theorem:

$$\|\theta(\mathbf{v}, \tau)\| \leq \|\theta(\mathbf{0}, \tau)\| + \sup_{\zeta\in[0,1]} \|d_{\mathrm{v}}\theta(\zeta\mathbf{v}, \tau)\| \, \|\mathbf{v}\|.$$

We note that each term of the series (1.48) clearly has the asymptotic

$$\mathbf{x}_k(\tau)e^{-k\beta\tau} = O(e^{-(kb+\sigma-\delta/2)\tau})$$

for arbitrary $\delta > 0$. Therefore

$$\theta(\mathbf{0}, \tau) = -\mathbf{u}'_K(\tau) + \mathbf{K}(\tau)\mathbf{u}_K(\tau) + \phi(\mathbf{u}_K(\tau), \tau) + \psi(\mathbf{u}_K(\tau), \tau).$$

We will estimate the quantity

$$d_{\mathbf{u}}\theta(\mathbf{v}, \tau) = d_{\mathbf{u}}\phi(\mathbf{u}_K(\tau) + \mathbf{v}, \tau) + d_{\mathbf{u}}\psi(\mathbf{u}_K(\tau) + \mathbf{v}, \tau),$$

whose order with respect to $\mathbf{u}$, like its minimum, is quadratic over the space of vector functions ϕ. Therefore $d_{\mathbf{u}}\phi(\mathbf{u}, \tau) = O(\|\mathbf{u}\|)$ and the asymptotic of this quantity is determined by the asymptotic of the first term of the series (1.48), whereby we may write

$$d_{\mathbf{u}}\phi(\mathbf{u}_k(\tau) + \mathbf{v}(\tau), \tau) = O(e^{-(\beta-\sigma-\delta/2)\tau}) = O(e^{-(b+\sigma-\delta/2)\tau}).$$

Over the space of vector functions ψ we have

$$d_{\mathbf{u}}\psi(\mathbf{u}_K(\tau) + \mathbf{v}(\tau), \tau) = O(e^{-\beta\tau}) = O(e^{-(b+2\sigma)\tau}).$$

The last asymptotics of the estimate are uniform in $\mathbf{v}$, in some small neighborhood of $\mathbf{v} = \mathbf{0}$, with respect to the standard norm on $\mathbb{R}^n$. Therefore

$$\theta(\mathbf{v}(\tau), \tau) = O(e^{-((K+1)b+\sigma-\delta/2)\tau})$$

for small $\delta > 0$.

Earlier we showed that the characteristic exponent of the matrix

$$\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)$$

equals σ, i.e. $\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau) = O(e^{(\sigma+\delta/2)\tau})$. It is therefore immediate that $\widehat{\theta}(\mathbf{w}(\tau), \tau)$ has the asymptotic

$$\widehat{\theta}(\mathbf{w}(\tau), \tau) = O(e^{-((K+1)b-\delta)\tau}).$$

Since the vector function $\widehat{\theta}(\mathbf{w}, \tau)$ is continuous with respect to all its arguments jointly, we may assume that $\widehat{\theta}$ establishes some mapping Θ of a neighborhood $U_{0,\Delta}$ of zero of the normed space $H_{0,\Delta}$ into itself, where:

$H_{0,\Delta}$ is the Banach space of vector functions $\mathbf{w}[\mathrm{T}, +\infty] \to \mathbb{R}^n$, continuous on the closed half-line $[\mathrm{T}, +\infty)$, for which the finite norm is

$$\|\mathbf{w}\|_{0,\Delta} = \sup_{[\mathrm{T},+y]} e^{\Delta\tau}\|\mathbf{w}(\tau)\|.$$

We consider too the much more restricted space $H_{1,\Delta}$:

$H_{1,\Delta}$ is the Banach space of vector functions $\mathbf{w} : [\mathrm{T}, +y]R^n$, continuous on the closed half-line $[\mathrm{T}, +y)$ along with their first derivatives, for which the finite norm is

$$\|\mathbf{w}\|_{1,\Delta} = \sup_{[\mathrm{T},+y]} e^{\Delta\tau}(\|\mathbf{w}(\tau)\| + \|\mathbf{w}'(\tau)\|),$$

and we likewise consider the bounded linear operator $\mathbf{L}$: $H_{1,\Delta} \to H_{0,\Delta}$, given by the formula

$$\mathbf{L} = \frac{d}{d\tau} - \mathbf{R}.$$

Lemma 1.3.4. *The operator* $\mathbf{L}$ *has a bounded inverse, whose norm does not depend on* T.

Proof. We consider the system of linear differential equations

$$\frac{d\mathbf{w}}{d\tau} - \mathbf{R}\mathbf{w} = \mathbf{h}, \quad \mathbf{h} \in H_{0,\Delta} \tag{1.54}$$

and find the unique particular solution of this system that is continuously differentiable on the half-line $[\mathrm{T}, +y)$ and satisfies the boundary condition $\mathbf{w}(+y) = \mathbf{0}$ as well as the inequality

$$\|\mathbf{w}\|_{1,\Delta} \leq C\|\mathbf{h}\|_{0,\Delta}, \tag{1.55}$$

where the constant $C > 0$ depends neither on $\mathbf{h} \in H_{0,\Delta}$ nor on the quantity $\mathrm{T} > 0$. The greatest lower bound of all such $C > 0$ will be the norm of the operator $\mathbf{L}^{-1}$.

The desired solution of the system (1.54) now assumes the form

$$\mathbf{w}(\tau) = -\exp(\mathbf{R}\tau)\int_{\tau}^{+y} \exp(\mathbf{R}\xi)\mathbf{h}(\xi)d\xi.$$

This solution satisfies the necessary condition for smoothness and has the required asymptotic. Because it satisfies the inequality $\Delta > -r_i$ for arbitrary i, the improper integral in these variables converges. Estimating separately in each coordinate, we obtain an explicit expression for the constant C in (1.55):

$$C = \max_i(1 + (|r_i| + 1)(\Delta + r_i)^{-1}).$$

The lemma is proved.

We rewrite the system of differential equations (1.53) in the form

$$\mathbf{w} = \mathcal{F}(\mathbf{w}), \tag{1.56}$$

where $\mathcal{F} = \mathbf{L}^{-1}\Theta$ maps the space $H_{0,\Delta}$ into itself.

We show that $\mathcal{F}$ is contractive on some small neighborhood $U_{0,\Delta}$ of zero of the space $H_{0,\Delta}$. For this we first note that, for large $\mathrm{T} > 0$, the inclusion $\mathcal{F}(U_{0,\Delta}) \subset U_{0,\Delta}$ holds.

In fact, we almost literally repeat the reasoning used for the estimate of $\widehat{\theta}(\tau, \mathbf{w}(\tau))$, obtaining the estimate

$$\|\Theta(\mathbf{w})\|_{0,\Delta} = O(e^{-b\mathrm{T}}) \text{ as } \mathrm{T} \to +y,$$

from which it follows that there exists a constant $L > 0$ such that

$$\|\mathcal{F}(\mathbf{w})\|_{(0,\Delta)} \leq \|\mathcal{F}(\mathbf{w})\|_{(1,\Delta)} \leq LCe^{-b\mathrm{T}},$$

where C is the norm of the operator $\mathbf{L}^{-1}$.

We estimate the difference:

$$\Theta(\mathbf{w}^{(1)}) - \Theta(\mathbf{w}^{(2)}) = \exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)(\theta(\mathbf{v}^{(1)}, \tau) - \theta(\mathbf{v}^{(2)}, \tau))$$

for $\mathbf{w}^{(1)}, \ \mathbf{w}^{(2)} \in U_{0,\Delta}$.

From the mean value theorem,

$$\|\theta(\mathbf{v}^{(1)}, \tau) - \theta(\mathbf{v}^{(2)}, \tau)\| \leq$$
$$\leq \sup_{\zeta \in [0,1]} \|d_{\mathrm{v}}\theta(\mathbf{v}^{(1)} + \zeta(\mathbf{v}^{(2)} - \mathbf{v}^{(1)}), \tau)\| \, \|\mathbf{v}^{(2)} - \mathbf{v}^{(1)}\|.$$

Because of the smallness—measured by the standard norm on $\mathbb{R}^n$—of the linear combination $\mathbf{v}^{(1)} + \zeta(\mathbf{v}^{(2)} - \mathbf{v}^{(1)})$, the *supremum* appearing in the above formula is a quantity with asymptotic $O(e^{-(b+\sigma-\delta/2)\tau})$ (see the argument for the corresponding order of $d_{\mathrm{v}}\theta$, given above). The norm of the matrix $\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)$ is a quantity of order $O(e^{(\sigma+\delta/2)\tau})$. Finally,

$$\|\mathbf{v}^{(2)}(\tau) - \mathbf{v}^{(1)}(\tau)\| \leq M(\tau)\|\mathbf{w}^{(2)}(\tau) - \mathbf{w}^{(1)}(\tau)\|,$$

where $M(\tau) = O(e^{\frac{\delta}{2}\tau})$ is some quantity dependent only on the "time" τ.

We can therefore assert that there is a constant $N > 0$, independent of τ, such that

$$\|\Theta(\mathbf{w}^{(1)}) - \Theta(\mathbf{w}^{(2)})\|_{0,\Delta} \leq Ne^{-(b-\frac{3\delta}{2})\mathrm{T}}\|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}\|_{0,\Delta},$$

from which the following estimate is immediate:

$$\|\mathcal{F}(\mathbf{w}^{(1)}) - \mathcal{F}(\mathbf{w}^{(2)})\|_{0,\Delta} \leq \|\mathcal{F}(\mathbf{w}^{(1)}) - \mathcal{F}(\mathbf{w}^{(2)})\|_{1,\Delta} \leq$$
$$\leq CNe^{-(b-\frac{3\delta}{2})\mathrm{T}}\|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}\|_{0,\Delta}.$$

From this it is clear that, for large $\mathrm{T} > 0$, the mapping $\mathcal{F}$ is contractive. Applying the Caciopoli-Banach principle [94] to Eq. (1.56), we obtain that the equation has a solution in $U_{0,\Delta}$, i.e. the mapping $\mathcal{F}$ has a fixed point. Inasmuch as the set of values of $\mathcal{F}$ in general lies in the space $H_{1,\Delta}$, the solution $\mathbf{w}(\tau)$ of (1.56) belongs to class $\mathbf{C}^1[\mathrm{T}, +y)$ and thus is a solution of the system of differential equations (1.53) with asymptotic $\mathbf{w}(\tau) = O(e^{-\Delta\tau})$. Since the right sides of (1.53) are infinitely differentiable vector functions, we have that $\mathbf{w}(\tau) \in \mathbf{C}^y[\mathrm{T}, +y)$. Returning to the variable $\mathbf{v}$, we obtain that $\mathbf{w}(\tau) = o(e^{-Kb\tau})$.

Lemma 1.3.3 is proved.

To complete the proof of Theorem 1.3.2, it is now necessary to return to the original variables $\mathbf{x}, t$. Thus the original system of equations (1.37) has an infinitely differentiable solution with the required asymptotic.

The theorem is proved.

Theorem 1.3.2, which we have just now proved in general, guarantees but one solution with the asymptotic that interests us. This solution can be produced constructively in the form of a series where, for finding the coefficients, we need at each step to solve a nonautonomous linear system of differential equations. In spite of this obvious advantage, the theorem doesn't even allow us to estimate the "size" of the set of such solutions, although the algorithm for constructing the series (1.45) indicates that the desired solution may depend on some number of free parameters. Below we establish a result, related to the Hadamard-Perron theorem [137], which guarantees the existence of some l–parameter family of solutions with the required asymptotic.

Theorem 1.3.3. *Suppose that the truncated system of equations (1.39) has a particular solution of form (1.36) such that, for $\beta > 0$, the full spectrum of system (1.41) and, for $\beta < 0$, the full spectrum of system (1.43), contains l negative characteristic exponents, the remaining ones being positive or zero, and such that the irregularity measure σ of the corresponding system satisfies the inequality*

$$\sigma < \min(\frac{|\beta|}{2}, -R), \tag{1.57}$$

where $R = \max(r_i : r_i < 0)$. Then (1.37) has an l–parameter family of particular solutions of the form

$$\mathbf{x}(\mathbf{c}, t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^{\gamma}(\gamma t) + o(1)) \text{ as } t^{\chi} \to \gamma \times \inf,$$

where $\mathbf{c} \in R^l$ is a vector of parameters.

The proof is in many ways conceptually similar to the proof of the preceding theorem. For the original system we take (1.47) and, as in the proof of the preceding theorem, we only look at the positive semi-quasihomogeneous case ($\chi = +1$). We actually only need prove the following assertion.

Lemma 1.3.5. *Suppose that on the right side of (1.47) the parameter β is positive, that the full spectrum of system (1.41) contains l negative characteristic exponents, the remaining ones being positive or zero, and that the irregularity measure σ of (1.41) satisfies the inequality (1.57). Then (1.47) has an l–parameter family of solutions that tend to $\mathbf{u} = \mathbf{0}$ as $\tau \to +\inf$.*

Proof. We change the dependent and independent variables

$$\mathbf{u}(\tau) = \varepsilon \mathbf{v}(\tau), \quad \tau_{\varepsilon} = \tau + 2\beta^{-1} \ln \varepsilon,$$

after which system (1.47) assumes the form

$$\mathbf{v}' = \mathbf{K}_{\varepsilon}(\tau_{\varepsilon})\mathbf{v} + \varepsilon(\phi(\mathbf{v}, \tau_{\varepsilon}, \varepsilon) + \psi(\mathbf{v}, \tau_{\varepsilon}, \varepsilon)), \tag{1.58}$$

where $\varepsilon > 0$ is some small parameter and where the prime now indicates differentiation with respect to the "new time" τ_{ε}, and the vector functions ϕ, ψ appearing on the right side of (1.58) are expressed by means of the preceding in the following form:

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} (\mathbf{v}, \tau_{\varepsilon}, \varepsilon) = \varepsilon^{-2} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} (\varepsilon \mathbf{v}, \tau + 2\beta^{-1} \ln \varepsilon).$$

We let $\theta(\mathbf{v}, \tau_\varepsilon, \varepsilon)$ denote the sum $\Phi(\mathbf{v}, \tau_\varepsilon, \varepsilon) + \Psi(\mathbf{v}, \tau_\varepsilon, \varepsilon)$. It is clear that this vector function is continuous in ε.

The matrix of the linear part of (1.58) is expressed by means of the Kovalevsky matrix in the following form:

$$\mathbf{K}_\varepsilon(\tau_\varepsilon) = \mathbf{K}(\tau_\varepsilon - 2\beta^{-1} \ln \varepsilon).$$

It is easy to see that the matrix $\mathbf{K}_\varepsilon(\tau_\varepsilon)$ gives rise to a linear system with exactly the same asymptotic properties as are enjoyed by system (1.41). For brevity we now drop the index ε in the variable τ_ε.

With the aid of the change of variables

$$\mathbf{v} = \mathbf{U}_\varepsilon(\tau) \exp(-\mathbf{R}\tau)\mathbf{w},$$

where $\mathbf{U}_\varepsilon(\tau)$ is the fundamental matrix of the system of linear differential equations with matrix $\mathbf{K}_\varepsilon(\tau)$, we get rid of nonautonominity in the linear part. As a result, we obtain a system resembling (1.53):

$$\mathbf{w}' = \mathbf{R}\mathbf{w} + \varepsilon\widehat{\theta}(\mathbf{w}, \tau, \varepsilon), \tag{1.59}$$

where

$$\widehat{\theta}(\mathbf{w}, \tau, \varepsilon) = \exp(\mathbf{R}\tau)\mathbf{U}_\varepsilon^{-1}(\tau)\theta(\mathbf{U}_\varepsilon(\tau) \exp(-\mathbf{R}\tau)\mathbf{w}, \tau, \varepsilon).$$

The phase space of system (1.59) decomposes into a direct sum

$$\mathbb{R}^n = E^{(s)} \oplus E^{(u,c)}$$

of subspaces invariant under the operator $\mathbf{R}$ and such that the spectrum of the restriction $\mathbf{R}|_{E^{(s)}} = \mathbf{R}^{(s)}$ is negative and the spectrum of $\mathbf{R}|_{E^{(u,c)}} = \mathbf{R}^{(u,c)}$ is nonnegative. The projections of $\mathbf{w}$ and $\widehat{\theta}$ onto the subspaces $E^{(s)}$ and $E^{(u,c)}$ are denoted, respectively, by $\mathbf{w}^{(s)}$, $\mathbf{w}^{(u,c)}$ and $\widehat{\theta}^{(s)}$, $\widehat{\theta}^{(u,c)}$.

We write the system of differential equations (1.59) in the form of a system of integral equations:

$$\begin{aligned} \mathbf{w}^{(s)} &= \varepsilon \exp\left(\mathbf{R}^{(s)}\tau\right) \left(\mathbf{c} + \int_0^\tau \exp\left(-\mathbf{R}^{(s)}\xi\right) \widehat{\theta}^{(s)}(\mathbf{w}, \xi, \varepsilon\right) d\xi \\ \mathbf{w}^{(u,c)} &= \varepsilon \exp\left(\mathbf{R}^{(u,c)}\tau\right) \left(\mathbf{c} + \int_\tau^\infty \exp\left(-\mathbf{R}^{(u,c)}\xi\right) \widehat{\theta}^{(u,c)}(\mathbf{w}, \xi, \varepsilon\right) d\xi, \end{aligned} \tag{1.60}$$

where $\mathbf{c} \in \mathbb{R}^l = E^{(s)}$ and $\|\mathbf{c}\| \leq 1$ is a vector of free parameters.

From (1.60) it is clear that the desired solution $\mathbf{w}(\tau)$ will satisfy the conditions $\mathbf{w}^{(s)}(0) = \mathbf{c}$, $\mathbf{w}^{(u,c)}(+\infty) = \mathbf{0}$. We will additionally require that $\mathbf{w}(\tau) = o(1)$ as $\tau \to +\infty$.

The problem of finding a solution for (1.60) may be regarded as the problem of finding a fixed point

$$\mathbf{w} = q_\varepsilon(\mathbf{w}), \tag{1.61}$$

where q_ε is the mapping of some small neighborhood $U_{0,\Delta}$ of the space $H_{0,\Delta}$into itself that was considered above, where now $\Delta = \sigma + 2\delta$, $\delta > 0$ is sufficiently small and where the vector functions $\mathbf{w}(\tau)$ are determined on the closed half-line $[0, +\infty)$.

Again Θ_ε denotes the mapping of $U_{0,\Delta}$ into the space $H_{0,\Delta}$ induced by the vector function $\widehat{\theta}(\mathbf{w}, \tau, \varepsilon)$.

Since

$$\mathbf{w}(\tau) = O(e^{-(\sigma+2\delta)\tau}) \text{ as } \tau \to +\inf,$$

it is clear that $\mathbf{v}(\tau) == O(e^{-(\sigma+\frac{3\delta}{2})\tau})$, whence

$$\phi(\mathbf{v}, \tau, \varepsilon) = O(e^{-(2\sigma+3\delta)\tau}) \text{ and } \psi(\mathbf{v}, \tau, \varepsilon) = O(e^{-\beta\tau}).$$

Because of the inequality (1.57), it is possible to choose δ small enough so that the following estimate holds:

$$\theta(\mathbf{v}, \tau, \varepsilon) = O(e^{-(2\sigma+3\delta)\tau}).$$

We can therefore use the asymptotic estimates obtained previously for the norm of the matrix $\exp(\mathbf{R}\tau)\mathbf{U}_\varepsilon^{-1}(\tau)$:

$$\widehat{\theta}(\mathbf{v}, \tau, \varepsilon) = O(e^{-(\sigma+\frac{5\delta}{2})\tau}).$$

These estimates are uniform in $\varepsilon, \mathbf{w}$ for small ε and for $\mathbf{w}$ in some small neighborhood $U_{0,\Delta}$ of the space $H_{0,\Delta}$. Thus there exists a constant $L > 0$ such that

$$\|\Theta_\varepsilon(\mathbf{w})\|_{0,\Delta} \leq L.$$

The mapping q_ε can be rewritten in the form $q_\varepsilon = \varepsilon\mathbf{P}\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}}$, where $\mathbf{P}$ is some linear mapping on the space $H_{0,\Delta}$, induced by the integral transformation of (1.60) and applied to $\widehat{\theta}(\mathbf{v}, \boldsymbol{\tau}_{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon})$.

Lemma 1.3.6. *The mapping* $\mathbf{P}$ *is continuous.*

Proof. We consider two arbitrary vector functions $\mathbf{h}^{(1)}, \mathbf{h}^{(2)} \in H_{0,\Delta}$ and set $\mathbf{w}^{(1)} = \mathbf{P}\mathbf{h}^{(1)}$, $\mathbf{w}^{(2)} = \mathbf{P}\mathbf{h}^{(2)}$. Since the Eq. (1.57) is satisfied, we can choose δ small enough so that the inequality $\Delta = \sigma + 2\delta < -R$ holds. Therefore

$$\begin{aligned}
&\left|\left(w^{(1)}\right)^i(\tau) - \left(w^{(2)}\right)^i(\tau)\right| \leq \left(e^{r_i\tau}\int_0^\tau e^{-(r_i+\Delta)\xi}d\xi\right)\left\|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\right\|_{0,\Delta} = \\
&= \left(-(\Delta + r_i)^{-1}e^{\Delta\tau}\left(1 - e^{(r_i+\Delta)\tau}\right)\right)\left\|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\right\|_{0,\Delta} \leq \\
&\leq e^{-\Delta\tau}\left|\Delta + r_i\right|^{-1}\left\|\mathbf{h}^{(1)} - \mathbf{h}^{(2)}\right\|_{0,\Delta}
\end{aligned}$$

for the i–th component, corresponding to the subspace $E^{(s)}$.

Then for the component corresponding to $E^{(u,c)}$ we have

$$\left|\left(w^{(1)}\right)^i(\tau)-\left(w^{(2)}\right)^i(\tau)\right| \leq \left(e^{r_i\tau}\int_\tau^\infty e^{-(r_i+\Delta)\xi}d\xi\right)\left\|\mathbf{h}^{(1)}-\mathbf{h}^{(2)}\right\|_{0,\Delta} =$$
$$= e^{-\Delta\tau}(\Delta+r_i)^{-1}\left\|\mathbf{h}^{(1)}-\mathbf{h}^{(2)}\right\|_{0,\Delta}$$

From these inequalities it follows that there exists a constant $C>0$ such that

$$\left\|\mathbf{w}^{(1)}-\mathbf{w}^{(2)}\right\|_{0,\Delta} \leq C\left\|\mathbf{h}^{(1)}-\mathbf{h}^{(2)}\right\|_{0,\Delta}.$$

The lemma is proved.

Consequently—since $\exp(\mathbf{R}^{(s)}\tau)\mathbf{c}\in H_{0,\Delta}$—the composition $\mathbf{P}\Theta_\varepsilon$ is bounded on $U_{0,\Delta}$. Since the mapping q_ε has the form $q_\varepsilon=\varepsilon\mathbf{P}\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}}$, a suitable choice of ε can be made so that $q_\varepsilon(U_{0,\Delta})\subset U_{0,\Delta}$.

We prove that q_ε is a contraction on $U_{0,\Delta}$ for small $\varepsilon>0$. We consider arbitrary functions $\mathbf{w}^{(1)},\mathbf{w}^{(2)}\in U_{0,\Delta}$ and estimate the difference in the norm:

$$\Theta_\varepsilon(\mathbf{w}^{(1)})-\Theta(\mathbf{w}^{(2)}) \leq \widehat{\theta}(\mathbf{v}^{(1)},\tau,\varepsilon)-\widehat{\theta}(\mathbf{v}^{(2)},\tau,\varepsilon).$$

Using the mean value theorem,

$$\left\|\theta(\mathbf{v}^{(1)},\tau,\varepsilon)-\theta(\mathbf{v}^{(2)},\tau,\varepsilon)\right\| \leq$$
$$\leq \sup_{\xi\in[0,1]}\left\|d_\nu\theta\left(\nu^{(1)}+\zeta\left(\nu^{(2)}-\nu^{(1)}\right),\tau,\varepsilon\right)\right\|\left\|\nu^{(2)}-\nu^{(1)}\right\|.$$

In view of the smallness under the standard norm on $\mathbb{R}^n$ of the linear combination $\mathbf{v}^{(1)}+\zeta(\mathbf{v}^{(2)}-\mathbf{v}^{(1)})$ and the fact that the inequality (1.57) is satisfied, an upper bound of the matrix $d_{\mathbf{v}}\boldsymbol{\theta}$ has the asymptotic $O(e^{-(\sigma+3\Delta/2)\tau})$. The norm of the matrix $\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)$ is a quantity of order $O(e^{(\sigma+3\Delta/2)\tau})$. Consequently, by studying the difference between the asymptotics of the quantities $\mathbf{v}(\tau)$ and $\mathbf{w}(\tau)$, we can write

$$\|\boldsymbol{\Theta}_\varepsilon(\mathbf{w}^{(1)})-\boldsymbol{\Theta}_\varepsilon(\mathbf{w}^{(2)})\|_{0,\Delta} \leq N\|\mathbf{w}^{(1)}-\mathbf{w}^{(2)}\|_{0,\Delta},$$

where $N>0$ is some constant not depending on $\mathbf{w}^{(1)},\mathbf{w}^{(2)}\in\mathfrak{U}_{0,\Delta}$, whence follows at once the estimate

$$\|\mathcal{F}_\varepsilon(\mathbf{w}^{(1)})-\mathcal{F}_\varepsilon(\mathbf{w}^{(2)})\|_{0,\Delta} \leq \varepsilon CN\|\mathbf{w}^{(1)}-\mathbf{w}^{(2)}\|_{0,\Delta}.$$

With a suitable choice of $\varepsilon>0$ we can obtain contractiveness for the mapping q_ε. From the Caccioppoli-Banach principle [137], applied to Eq. (1.61), we obtain that this equation has a solution in $\mathfrak{U}_{0,\Delta}$, i.e. the mapping $\mathcal{F}_\varepsilon$ has a fixed point and the system of integral equations (1.60) has a solution with asymptotic $\mathbf{w}(\tau)=O(e^{-\Delta\tau})$. Since in general the operator $\mathbf{P}$ increases the order of smoothness by one, the solution $\mathbf{w}(\tau)$ of (1.61) belongs to class $\mathbf{C}^1[0,+\infty)$, i.e. it constitutes a solution to the system of differential equations (1.59). In all of this we are not speaking of an isolated solution, but of a family of solutions tending to zero as $\tau \to +\infty$.

Lemma 1.3.5 is proved.

Returning to the original variables $\mathbf{x}, t$ we obtain that the original system of differential equations (1.37) has an l–parameter family of particular solutions with the required asymptotic.

Theorem 1.3.3 is proved.

We finish this section with a series of remarks.

Remark 1.3.1. The classical Hadamard-Perron theorem [137] asserts that exponential trajectories "sweep out" some set that has, in the neighborhood of a critical point, the structure of a smooth manifold. Even in the autonomous case, the set of solutions with generalized power asymptotic—whose existence is guaranteed by the theorem just proved—don't form a smooth manifold.

Remark 1.3.2. In proving the preceding theorem, we in fact proved that, in the new "logarithmic" time τ, the "perturbed" $\mathbf{u}(\tau)$, which generates a solution (1.36) of the truncated system, has asymptotic $\mathbf{u}(\tau) = O\left(e^{-\left(\sigma+\frac{3}{2}\delta\right)\tau}\right)$ for arbitrarily small $\delta > 0$, which implies formally that, with a reduced irregularity measure σ, this asymptotic worsens and that, in the case of the regularity of the system of first approximation, (1.41) can only guarantee the very weak asymptotic $O\left(e^{-\left(\frac{3}{2}\delta\right)\tau}\right)$ for $\mathbf{u}(\tau)$. Nonetheless, by carrying out a very similar proof for the case of a regular system (1.41), it can be shown that $\mathbf{u}(\tau)$ has an asymptotic of a much higher order, specifically $O\left(e^{-(\beta-\delta)\tau}\right)$.

Remark 1.3.3. It can happen that some components of the solution $\mathbf{x}^{\gamma}(t)$ generated turn out to equal zero. Then, in these components the principal terms of the asymptotic will be determined in the autonomous case by the eigenvalues of the Kovalevsky matrix. We will come upon a similar situation when we consider Example 1.4.5 of the following section.

Remark 1.3.4. From the very beginning of this section we have assumed that the right side of system (1.37) is a semi-quasihomogeneous vector field, provided the time appearing in them is regarded as a parameter (i.e. t in the right sides of (1.37) isn't changed under the action of the group (1.11)). However, practically all the results obtained remain valid even in much more general situations. Let the right side of (1.37) have the following form:

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}_q(\mathbf{x}, t) + \mathbf{f}^*(\mathbf{x}, t) + \mathbf{f}^{**}(\mathbf{x}, t),$$

where $\mathbf{f}_q(\mathbf{x}, t)$ is a quasihomogeneous vector field, $\mathbf{f}^*(\mathbf{x}, t)$ transforms, under the action of the group (1.11), into a power series in μ^{β} without free term, and where $\mathbf{f}^{**}(\mathbf{x}, t)$ becomes a quantity of order $O\left(\mu^{M\beta}\right)$ for some sufficiently large M (as before, the time t in the right hand side does not change). In this, the result of the transformation of $\mathbf{f}^{**}(\mathbf{x}, t)$ can depend on μ in a rather complicated way (it can, for example, contain logarithms and periodic or quasiperiodic functions of μ).

The presence of the $\mathbf{f}^{**}(\mathbf{x}, t)$ term can hinder attaining the construction of a formal series (1.45). However, this does not prevent the proof of existence of a particular solution of the original system with required asymptotic.

1.4 Examples

We now apply the methods that have been developed in the preceding sections to some concrete examples.

Example 1.4.1. Following the article [60], we investigate the problem of the stability of a critical point and the existence of asymptotic solutions of a multiple-dimensional smooth system of differential equations, whose linear part represents a Jordan decomposition with zero diagonal (Lyapunov's problem).

We write this system of differential equations in the following way:

$$\dot{x}^i = x^{i+1} + \ldots, \quad i = 1, \ldots, n-1, \quad \dot{x}^n = a(x^1)^2 + \ldots, \tag{1.62}$$

where the dots in the equations indicate the presence of nonlinear terms, of which only the monomial $a(x^1)^2$ remains in the final equation. It's easy to see that the chosen system is quasihomogeneous of degree $\mathrm{q} = 2$ in the sense of Definition 1.2.1, with quasihomogeneity indices determined by the integral diagonal matrix $\mathbf{S} = \mathrm{diag}(n, n+1, \ldots, 2n-1)$. It can be shown that the quasihomogeneous system under consideration is generated by the positive faces of the Newton polytope for the system (1.62), so that the system (1.62) is positive semi-quasihomogeneous ($\chi = +1$). If the coefficient $a \neq 0$, then the quasihomogeneous truncation has the particular asymptotic solution

$$\mathbf{x}^-(t) = (-t)^{-\mathbf{S}}\mathbf{x}_0^-, \quad \text{or} \quad x^{-i}(t) = \frac{x_0^{-i}}{(-t)^{n+i-1}}, \quad i = 1, \ldots, n,$$

where

$$x_0^{-i} = \frac{(2n-1)!(n+i-2)!}{a((n-1)!)^2}, \quad i = 1, \ldots, n,$$

from which it follows that the full system has the particular asymptotic solution $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$. This fact indicates the instability of the critical point considered. For the two-dimensional system ($n = 2$) the result obtained represents Lyapunov's theorem [133].

It is possible to show that the truncated system also has a "positive" particular solution $\mathbf{x}^+(t) = t^{-\mathbf{S}}\mathbf{x}_0^+$—which implies as well the existence of an asymptotic solution of the full system that approaches the critical point as $t \to +\infty$. This solution of the truncated system can be found immediately. On the other hand, its existence follows from part (a) of Lemma 1.1.1. Inasmuch as the right sides of the truncated system are invariant with respect to the substitution $x^1 \mapsto -x^1$, the degree of the Gauss map Γ is even, so that Γ has a fixed point for which the antipode is also a fixed point.

Suppose that in the system (1.62) there is the additional degeneracy ($a = 0$). The two-dimensional case was investigated in detail by Lyapunov [133], so that we will concentrate on the case $n \geq 3$.

We rewrite system (1.62) in the following form:

$$\begin{gathered} \dot{x}^i = x^{i+1} + \dots, \quad i = 1, \dots, n-2 \\ \dot{x}^{n-1} = x^n + b(x^1)^2 + \dots, \quad \dot{x}^n = 2cx^1x^2 + \dots \end{gathered} \tag{1.63}$$

Here the dots in the equations again indicate the presence of nonlinear terms, from which the monomial $b(x^1)^2$ is singled out in the penultimate equation, and $2cx^1x^2$ is singled out in the last equation. The system chosen is quasihomogeneous of degree q $= 2$ with matrix of indices $\mathbf{S} = \operatorname{diag}(n-1, \dots, 2n-2)$. If $b + c \neq 0$, then this system has a particular asymptotic solution of the form

$$\mathbf{x}^-(t) = (-t)^{-\mathbf{S}}\mathbf{x}_0^-, \quad \text{or} \quad x^{-i}(t) = \frac{x_0^{-i}}{(-t)^{n+i-2}}, \quad i = 1, \dots, n,$$

where

$$x_0^{-i} = \frac{(2n-3)!(n+i-3)!}{(b+c)((n-2)!)^2}, \quad i = 1, \dots, n-1,$$
$$x_0^{-n} = c\left(\frac{(2n-3)!}{(b+c)(n-2)!}\right)^2.$$

It is easy to show that the quasihomogeneous truncation is determined by the positive faces of the Newton polytope for system (1.63), so that (1.63) has an asymptotic solution $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$, which guarantees instability. It is also possible to show the existence of a solution tending toward $\mathbf{x} = \mathbf{0}$ as $t \to +\infty$.

We note that in both cases the solutions that are asymptotic as $t \to +\infty$ can be written in the following form:

$$\mathbf{x}(t) = t^{-\mathbf{S}} \sum_{k=0}^{\infty} \mathbf{x}_k(\ln t) t^{-k}.$$

Here the $\mathbf{x}_k(\cdot)$ are certain vector polynomials.

Example 1.4.2. The so-called logistical system of equations offers the simplest example of a situation where $\chi = -1$:

$$\dot{N}^i = N^i \left(k_i + b_i^{-1} \sum_{p=1}^{n} a_p^i N^p\right), \quad i = 1, \dots, n. \tag{1.64}$$

With the help of system (1.64) we can describe the interactions of diverse population types in an ecosystem. Here $N^i(t)$ is the number of individuals in the population of i–th type at time t, $(a^i_p)^n_{i,p=1}$ is a constant matrix (if $a^i_p > 0$, then this indicates that the i–th species is growing on account of the p–th species; in the opposite case the i–th species decreases on account of the p–th species), k_i is the difference between the birth and death rates of the i–th species when left to itself, the $b_i > 0$ are parameters characterizing the fact that the reproduction of one of the "predators" is associated with death in one or more of the prey populations. It makes sense to consider a real system of type (1.64) only in the first orthant ($N^i \geq 0$, $i = 1, \dots, n$). The properties of solutions of system (1.64) were first considered by Volterra [189], and the system (1.64) was later used for modeling other important applied problems. It was originally proposed in [189] that the matrix (a^i_p) be skew-symmetric; at the present time there are other models, where this requirement has been removed. For example, in the article [95], where system (1.64) was applied to the analysis of the dynamics of competing subsystems of production (job systems), it was proposed that off-diagonal elements be positive: $a^i_p > 0$, $i, p = 1, \dots, n$, $i \neq p$, and the diagonal elements be negative: ($a^i_i < 0$).

Removing the linear terms from (1.64), we get a quadratic homogeneous truncation ($\mathbf{S} = \mathbf{E}, \mathrm{q} = 2$),

$$\dot{N}^i = b_i^{-1} N^i \sum_{p=1}^{n} a^i_p N^p, \quad i = 1, \dots, n, \tag{1.65}$$

which is clearly selected by the negative faces of the Newton polytope for system (1.64).

System (1.65) has a particular solution of ray type,

$$\mathbf{N}^+(t) = t^{-1}\mathbf{N}_0^+, \quad \mathbf{N} = (N^1, \dots, N^n),$$

if the algebraic system of linear equations

$$\sum_{p=1}^{n} a^i_p N_0^{+p} = -b_i$$

is solved for $\mathbf{N}_0^+ = (N_0^{+1}, \dots, N_0^{+n})$.

Then the full system of equations has a particular solution with asymptotic decomposition

$$\mathbf{N}(t) = t^{-1} \sum_{k=0}^{\infty} \mathbf{N}_k (\ln t) t^k.$$

In order that the given particular solution be positive, it is sufficient that there be a positive solution for the above linear system.

The particular solution we have found has the asymptotic $\mathbf{N}(t) = O(t^{-1})$ as $t \to +0$. Since the right side of the chosen truncation is invariant with respect to the substitution $\mathbf{N} \mapsto -\mathbf{N}$, there likewise exists a particular solution with asymptotic $\mathbf{N}(t) = O((-t)^{-1})$ as $t \to -0$.

Example 1.4.3. Rössler's system [153]. We will find particular solutions with nonexponential asymptotic for a nonlinear system for which a chaotic attractor had previously been detected numerically. We will show that there are logarithmic terms in the corresponding expansions of these solutions, i.e. that the given system doesn't pass the Painlevé test.

This third order system has the form

$$\dot{x} = -(y + z), \quad \dot{y} = x + ay, \quad \dot{z} = a + xz - bz, \tag{1.66}$$

where a, b are real parameters.

There are many ways of choosing a quasihomogeneous truncated system for (1.66) and thus also of constructing nonexponential asymptotic solutions of the full system.

We subject system (1.66) to the action of the quasihomogeneous group of dilations of type (1.7):

$$x \mapsto \mu^{g_x} x, \quad y \mapsto \mu^{g_y} y, \quad z \mapsto \mu^{g_z} z, \quad t \mapsto \mu^{-1} t,$$

upon which this system (1.66) assumes the form

$$\begin{aligned}
\dot{x} &= -\mu^{g_z - g_x - 1} y - \mu^{g_z - g_x - 1} z, \\
\dot{y} &= \mu^{g_x - g_y - 1} x + a\mu^{-1} z, \\
\dot{z} &= a\mu^{-g_z - 1} + \mu^{g_x - 1} xz - b\mu^{-1} z.
\end{aligned}$$

The three numbers g_x, g_y, g_z are the diagonal elements of the diagonal matrix $\mathbf{G}$. It is clear that $\chi = -1$.

We restrict ourselves to cases where the elements of the matrix $\mathbf{G}$ are integers. In order to preserve the unique nonlinear term in the quasihomogeneous truncation, we must set $g_x = 1$. Then the possible values of the other two indices will be: $g_y \in \{0, 1, 2\}$, $g_z \in \{-1, 0, 1, 2\}$. Upon calculation of all principal possible combinations we enumerate the nontrivial particular solutions of "ray" type:

$$\begin{array}{lll}
\text{(a)} & \mathbf{G} = \operatorname{diag}(1, 0, -1), & x = 0,\ y = \eta,\ z = at, \\
\text{(b)} & \mathbf{G} = \operatorname{diag}(1, 0, 2), & x = 0,\ y = \eta,\ z = 0, \\
\text{(c)} & \mathbf{G} = \operatorname{diag}(1, 1, -1), & x = 0,\ y = 0,\ z = at, \\
\text{(d)} & \mathbf{G} = \operatorname{diag}(1, 1, 0), & x = 0,\ y = 0,\ z = \zeta, \\
\text{(e)} & \mathbf{G} = \operatorname{diag}(1, 1, 2), & x = -2t^{-1},\ y = 0,\ z = -2t^{-2}, \\
\text{(f)} & \mathbf{G} = \operatorname{diag}(1, 2, -1), & x = 0,\ y = 0,\ z = at, \\
\text{(g)} & \mathbf{G} = \operatorname{diag}(1, 2, 2), & x = -2t^{-1},\ y = 0,\ z = -2t^{-2}.
\end{array}$$

Cases (b) and (d) correspond to particular solutions of the full system (1.66), represented by the series

$$x(t) = \sum_{k=0}^{\infty} x_k t^k, \quad y(t) = \sum_{k=0}^{\infty} y_k t^k, \quad z(t) = \sum_{k=0}^{\infty} z_k t^k,$$

where the coefficients x_k, y_k, z_k don't depend on $\ln t$ and are polynomial functions of three free parameters ξ, η, ζ, and where we can take $\xi = x_0$, $\eta = y_0$, $\zeta = z_0$.

Cases (a), (c) and (f) correspond to particular solutions representing subfamilies of the considered family for $\zeta = 0$, which implies that $z_1 = a$.

We can also assert that the series constructed converge for arbitrary finite ξ, η, ζ in some small complex neighborhood of $t = 0$. This assertion is a simple consequence of Cauchy's theorem on the holomorphic dependence on time and initial conditions (see e.g. [42]).

Cases (e) and (g) yield particular solutions with the asymptotic expansions

$$\begin{aligned} x(t) &= t^{-1} \sum_{k=0}^{\infty} x_k(\ln t) t^k, \\ y(t) &= t^{-2} \sum_{k=0}^{\infty} y_k(\ln t) t^k, \\ z(t) &= t^{-2} \sum_{k=0}^{\infty} z_k(\ln t) t^k, \end{aligned}$$

where $x_0 = -2$, $y_0 = 0$, $z_0 = -2$ and where the logarithmic terms are unavoidable.

This indicates that Rössler's system doesn't have the Painlevé property and provides indirect confirmation of its chaotic nature.

Example 1.4.4. Following [116], we consider Hill's problem [80]. It is written as a Hamiltonian system of equations, whose Hamiltonian function is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + p_x y - p_y x - x^2 + \frac{1}{2} 12 y^2 - (x^2 + y^2)^{-1/2}. \tag{1.67}$$

The corresponding differential equations

$$\begin{aligned} \dot{p}_x &= p_y + 2x - x(x^2 + y^2)^{-\frac{3}{2}}, \quad & \dot{x} &= p_x + y, \\ \dot{p}_y &= -p_x - y - y(x^2 + y^2)^{-\frac{3}{2}}, \quad & \dot{y} &= p_y - x \end{aligned} \tag{1.68}$$

describe the planar motion of a satellite of small mass, e.g. a moon, in the gravitational field of two bodies, the mass of one of which is small in comparison with the mass of the other. for example the earth and the sun. A detailed statement of the problem can be found in the monograph [46]. It is interesting to note that, as was shown by Spring and Waldvogel [179], Eq. (1.68) also approximately describes the joint motion of two satellites of a massive attractive body in close orbit.

We introduce the new auxiliary variable $s = (x^2 + y^2)^{-1/2}$, converting (1.68) into a polynomial system of differential equations of the fifth order:

$$\begin{array}{ll} \dot{p}_x = p_y + 2x - xs^3, & \dot{x} = p_x + y, \\ \dot{p}_y = -p_x - y - ys^3, & \dot{y} = p_y - x, \\ \dot{s} = -xp_xs^3 - yp_ys^3. & \end{array} \tag{1.69}$$

Introducing for the phase variables the quasihomogeneous scale

$$\begin{array}{ccc} p_x \mapsto \mu^{g_{p_x}} p_x, & x \mapsto \mu^{g_x}, & p_y \mapsto \mu^{g_{p_y}} p_y, \\ y \mapsto \mu^{g_y}, & s \mapsto \mu^{g_s} s, & t \mapsto \mu^{-1} t, \end{array}$$

we find various truncations.

If we choose

$$g_{p_x} = -2/3, \quad g_x = -5/3, \quad g_{p_y} = 1/3, \quad g_y = -2/3, \quad g_s = 2/3,$$

then, under the action of the transformation indicated, the system of equations (1.69) takes the form

$$\begin{array}{ll} \dot{p}_x = \ p_y + 2\mu^{-2}x - xs^3, & \dot{x} = p_x + y, \\ \dot{p}_y = -\mu^{-2}p_x - \mu^{-2}y - ys^3, & \dot{y} = p_y - \mu^{-2}x, \\ \dot{s} \ = -\mu^{-2}xp_xs^3 - yp_ys^3. & \end{array} \tag{1.70}$$

From this it is clear that, with respect to the "scale" introduced, the system (1.69) is negative semi-quasihomogeneous. Setting $\mu = \infty$, we get a truncated quasihomogeneous system having the particular solution

$$p_x = p_{x_0}t^{2/3}, \ x = x_0t^{5/3}, \ p_y = p_{y_0}t^{-1/3}, \ y = y_0t^{2/3}, \ s = s_0t^{-2/3},$$

where

$$\begin{array}{c} p_{x_0} = \pm\frac{2}{3}\left(\frac{9}{2}\right)^{1/3}, \quad x_0 = \pm\left(\frac{9}{2}\right)^{1/3}, \\ p_{y_0} = \pm\frac{2}{3}\left(\frac{9}{2}\right)^{1/3}, \quad y_0 = \pm\left(\frac{9}{2}\right)^{1/3}, \quad s_0 = \left(\frac{2}{9}\right)^{1/3}. \end{array}$$

In accordance with Theorem 1.1.2, the Hamiltonian system of equations with Hamiltonian (1.67) has a particular solution admitting the asymptotic expansion

$$\begin{array}{ll} p_x(t) = t^{2/3}\sum_{k=0}^{\infty} p_{x_k}t^{k/3}, & x(t) = t^{5/3}\sum_{k=0}^{\infty} x_kt^{k/3}, \\ p_y(t) = t^{-1/3}\sum_{k=0}^{\infty} p_{y_k}t^{k/3}, & y(t) = t^{2/3}\sum_{k=0}^{\infty} y_kt^{k/3}. \end{array} \tag{1.71}$$

But if we choose

$$g_{p_x} = 1/3, \quad g_x = -2/3, \quad g_{p_y} = -2/3, \quad g_y = -5/3, \quad g_s = 2/3,$$

then (1.69) transforms to

$$
\begin{aligned}
\dot{p}_x &= \mu^{-2} p_y + 2\mu^{-2} x - x s^3, & \dot{x} &= p_x + \mu^{-2} y, \\
\dot{p}_y &= -p_x - \mu^{-2} y - y s^3, & \dot{y} &= p_y - x, \\
\dot{s} &= -x p_x s^3 - \mu^{-2} y p_y s^3.
\end{aligned}
\tag{1.72}
$$

For $\mu = \infty$ we get a quasihomogeneous truncated system with the particular solution

$$p_x = p_{x_0} t^{-1/3}, \quad x = x_0 t^{2/3}, \quad p_y = p_{y_0} t^{2/3}, \quad y = y_0 t^{5/3}, \quad s = s_0 t^{-2/3},$$

where now

$$
\begin{gathered}
p_{x_0} = \pm\tfrac{2}{3}\left(\tfrac{9}{2}\right)^{1/3}, \quad x_0 = \pm\left(\tfrac{9}{2}\right)^{1/3}, \\
p_{y_0} = \mp\tfrac{2}{3}\left(\tfrac{9}{2}\right)^{1/3}, \quad y_0 = \mp\left(\tfrac{9}{2}\right)^{1/3}, \quad s_0 = \left(\tfrac{2}{9}\right)^{1/3}.
\end{gathered}
$$

Consequently, the full system of equations (1.68) has a particular solution with asymptotic expansions

$$
\begin{aligned}
p_x(t) &= t^{-1/3} \sum_{k=0}^{\infty} p_{x_k} t^{k/3}, & x(t) &= t^{2/3} \sum_{k=0}^{\infty} x_k t^{k/3}, \\
p_y(t) &= t^{2/3} \sum_{k=0}^{\infty} p_{y_k} t^{k/3}, & y(t) &= t^{5/3} \sum_{k=0}^{\infty} y_k t^{k/3}.
\end{aligned}
\tag{1.73}
$$

Application of the algorithm in its general form for constructing formal asymptotics of the type (1.14) is explained in Sect. 1.1 and provides polynomial dependency on $\ln t$ for the coefficients of the expansions (1.71) and (1.73). But straightforward (although tedious) calculations show that p_{x_k}, x_k, p_{y_k}, y_k are constant. This happens for the following reasons. Inasmuch as the system (1.69) is negative semi-quasihomogeneous, so the logarithms may appear in stages, the k-th of which "resonates" with the positive eigenvalues of the Kovalevsky matrix, i.e. $-k\beta = \rho_i$, where the ρ_i are roots of the characteristic equation

$$\det(\mathbf{K} - \rho\mathbf{E}) = 0.$$

The Kovalevsky exponents for all the cases considered are as follows:

$$\rho_{1,2} = -1, \quad \rho_{3,4} = -4/3, \quad \rho_5 = 2/3,$$

where here $\beta = -1/3$. This indicates that the logarithms can't appear before the second stage. But from Eqs. (1.70) and (1.72) it is clear that the nonzero free terms $\mathbf{\Phi}_k$ in the equations for determining the coefficients appear only at the sixth stage.

The trajectories for Hill's problem corresponding to the particular solutions (1.68), with asymptotic expansions (1.71) and (1.73), are so-called collision trajectories. On these trajectories there occur collisions of the less massive body and the satellite (Earth and Moon) after a finite time (which depends on the chosen reference frame as $t \to 0$). The algorithm explained in Sect. 1.1 allows for the recurrent determination of the coefficients of these expansions and thus for the constructive determination of these collision trajectories in real time. Previously, in considering collisions in Hill's problem, the equations of motion were subjected to regularizing changes of variable that had been introduced by G.D. Birkhoff [16]. A contemporary treatment of this problem is given in the monograph [8].

Example 1.4.5. Generalized Henon-Heiles [78] system. This example is considered in detail in the articles [36, 37].

We consider the motion of a mechanical system with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + Dx^2y - \frac{C}{3}y^3. \tag{1.74}$$

The corresponding system of equations has the form

$$\begin{aligned} \dot{p}_x &= -x - 2Dxy, & \dot{x} &= p_x, \\ \dot{p}_y &= -y - Dx^2 + Cy^2, & \dot{y} &= p_y. \end{aligned} \tag{1.75}$$

We subject system (1.75) to the natural quasihomogeneous group of transformations

$$p_x \mapsto \mu^{g_x+1} p_x, \; x \mapsto \mu^{g_x}, \; p_y \mapsto \mu^{g_y+1} p_y, \; y \mapsto \mu^{g_y}, \; t \mapsto \mu^{-1} t,$$

after which (1.80) assumes the form

$$\begin{aligned} \dot{p}_x &= -\mu^{-1}x - 2\mu^{g_y-2}Dxy, & \dot{x} &= p_x, \\ \dot{p}_y &= -\mu^{-1}x - \mu^{2g_x-g_y-2}Dx^2 + \mu^{g_y-2}Cy^2, & \dot{y} &= p_y. \end{aligned} \tag{1.76}$$

From (1.76) it is clear that $\chi = 1$, so we seek the asymptotics as $t \to 0$.

To keep the nonlinear term in the first equation of (1.76) we need to set $g_y = 2$. The choices for g_x can lead to quite different variants. If our goal is to keep all the nonlinear terms in (1.76), then we need to set $g_x = 2$. Setting $\mu = \infty$, we obtain a truncated system with the particular solution:

$$p_x = p_{x_0} t^{-3}, \quad x = x_0 t^{-2}, \quad p_y = p_{y_0} t^{-3}, \quad y = y_0 t^{-2},$$

where

$$p_{x_0} = -2x_0, \; x_0 = \pm\frac{3}{D}(2 + \delta^{-1})^{1/2}, \; p_{y_0} = -2y_0, \; y_0 = -\frac{3}{D}, \; \delta = \frac{D}{C}.$$

The parameter $\delta \in \mathbb{R}$ will play an essential role in what follows.

We investigate the case where the expansions of solutions of (1.75), whose existence is guaranteed by Theorem 1.1.2, do not contain logarithmic time, so that they have the form

$$p_x(t) = t^{-3} \sum_{k=0}^{\infty} p_{x_k} t^k, \quad x(t) = t^{-2} \sum_{k=0}^{\infty} x_k t^k,$$
$$p_y(t) = t^{-3} \sum_{k=0}^{\infty} p_{y_k} t^k, \quad y(t) = t^{-2} \sum_{k=0}^{\infty} y_k t^k, \tag{1.77}$$

where $p_{x_k} = (k-2)x_k$, $p_{y_k} = (k-2)y_k$.

We note that in view of the invertibility of the system of equations with Hamiltonian (1.74), all the coefficients in (1.77) with odd indices reduce to zero.

It is not difficult to compute the eigenvalues of the Kovalevsky matrix, which in the case considered equal

$$\rho_1 = -1, \quad \rho_2 = 6, \quad \rho_{3,4} = \frac{5}{2} \pm \frac{1}{2}\left(1 - 24(1+\delta^{-1})\right)^{1/2}.$$

Thus logarithms can appear either after the third stage ($k = 6$) or from the resonances

$$\frac{5}{2} + \frac{1}{2}\left(1 - 24(1+\delta^{-1})\right)^{1/2} = 2l = k. \tag{1.78}$$

Calculations that were carried out with the aid of symbolic computation show that, with $k = 6$, logarithms do not appear. Therefore, in the absence of the resonances (1.78), the expansions (1.77) actually do not contain logarithmic time.

If we don't require keeping the term quadratic in x in the second equation of (1.76), in selecting the truncation, then we must choose $g_x < 2$. In order that the system (1.75) remain semi-quasihomogeneous in the sense of the usual definition it is also necessary that g_x be rational.

The analysis carried out in the paper [36] shows that the system of equations (1.75) has a particular solution with principal terms that have asymptotic expansions

$$p_x(t) \sim p_{x_0} t^{\Delta_\pm(\delta)-1}, \; x(t) \sim x_0 t^{\Delta_\pm(\delta)}, \; p_y(t) \sim p_{y_0} t^{-3}, \; y(t) \sim y_0 t^{-2}$$

as $t \to 0$, where $p_{x_0} = \Delta_\pm(\delta)c$, $x_0 = c$, $p_{y_0} = \frac{12}{C}$, $y_0 = \frac{6}{C}$, where c is some arbitrary constant and $\Delta_\pm(\delta) = \frac{1}{2} \pm \frac{1}{2}(1-48\delta)^{1/2}$. The quantities $\Delta_\pm(\delta)$ can of course be irrational and, for $\delta > \frac{1}{48}$, even complex. This indicates that this kind of solution can't be constructed with the aid of the algorithm of Theorem 1.1.2, nor indeed with any quasihomogeneous truncation.

What is the nature of the solution constructed in [36]? For arbitrary rational $g_x < 2$, the truncation

$$\dot{p}_x = -2Dxy, \quad \dot{x} = p_x, \quad \dot{p}_y = Cy^2, \quad \dot{y} = p_y$$

of the system has the particular solution

$$p_x \equiv 0, \quad x \equiv 0, \quad p_y = -\frac{12t^{-3}}{C}, \quad y = \frac{6t^{-2}}{C},$$

generated by the given quasihomogeneous structure.

We easily convince ourselves that the Kovalevsky exponents are here equal to

$$\rho_1 = -1, \quad \rho_2 = 6, \quad \rho_{3,4} = \Delta_{\pm}(\delta) + g_x.$$

Subsequent application of Theorem 1.3.1 makes it possible to establish existence of particular solutions of (1.75) with the indicated asymptotic.

Example 1.4.6. The Painlevé equations [68, 71, 87] can contain examples of nonautonomous semi-quasihomogeneous systems. It is claimed that a differential equation doesn't have movable critical points if the critical points of its solutions don't fill some region in the complex plane. Painlevé and Gambier classified equations of the form

$$\ddot{x} = R(\dot{x}, x, t),$$

not having moving singularities, where R is a rational function of x, $\dot{x}$, with coefficients meromorphic in t. Equations satisfying these requirements are often called equations of class P. A list of 50 equations was found such that each equation of class P can be obtained from one of the equations in the list with the aid of a certain holomorphic diffeomorphism satisfying some supplementary properties that we won't discuss here. Of these 50 equations, 44 are integrable by quadratures or can be transformed to an equation of the type

$$Q(\dot{x}, x, t) = 0,$$

where Q is a polynomial in $\dot{x}, x$ with meromorphic coefficients. The remaining six are called Painlevé equations. We enumerate them:

$$\text{I.} \qquad \ddot{x} = 6x^2 + t \tag{1.79}$$

$$\text{II.} \qquad \ddot{x} = 2x^3 + tx + a \tag{1.80}$$

$$\text{III.} \qquad \ddot{x} = \dot{x}^2 x^{-1} + e^t(ax^2 + b) + e^{2t}(cx^3 + dx^{-1}) \tag{1.81}$$

$$\text{IV.} \qquad \ddot{x} = \frac{1}{2}\dot{x}^2 x^{-1} + \frac{3}{2}x^3 + 4tx^2 + 2(t^2 - a)x + bx^{-1} \tag{1.82}$$

$$\text{V.} \qquad \ddot{x} = \dot{x}^2\left(\frac{1}{2x} + \frac{1}{x-1}\right) - \frac{\dot{x}}{t} + \frac{(x-1)^2}{t^2}\left(ax + \frac{b}{x}\right) + \\ + c\frac{x}{t} + d\frac{x(x+1)}{x-1} \tag{1.83}$$

$$\text{VI.} \qquad \ddot{x} = \frac{\dot{x}^2}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right) - \dot{x}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right) + \\ + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left(a + b\frac{1}{x^2} + c\frac{t-1}{(x-1)^2} + d\frac{t(t-1)}{(x-t)^2}\right) \tag{1.84}$$

All solutions of the first four Painlevé equations are meromorphic functions. The solutions of the fifth have logarithmic branch points at $t = 0$, $t = \infty$, as do those of the sixth at: $t = 0$, $t = 1$, $t = \infty$.

We investigate the form of the expansions of solutions of the Painlevé equations, obtained with the aid of the algorithms described above.

Written in the form of systems of two equations of first order,

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = 6x^2 + t, \\ \dot{x} &= y, \quad \dot{y} = 2x^3 + tx + a, \end{aligned}$$

the first and second Painlevé equations (1.79) and (1.80) are negative semi-quasihomogeneous with respect to the structure generated by the corresponding matrices $\mathbf{G} = \operatorname{diag}(2, 3)$ and $\mathbf{G} = \operatorname{diag}(1, 2)$. The parameter β in both cases can be taken equal to -1.

Their quasihomogeneous truncations

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = 6x^2, \\ \dot{x} &= y, \quad \dot{y} = 2x^3 \end{aligned}$$

have the obvious solutions

$$\begin{aligned} x &= \tfrac{1}{t^2}, \quad y = -\tfrac{2}{t^3}, \\ x &= \tfrac{1}{t}, \quad y = -\tfrac{1}{t^2}. \end{aligned}$$

The eigenvalues of the Kovalevsky matrix equal -1, 6 for the first Painlevé equation and -1, 4 for the second. A detailed analysis shows that logarithms don't appear in the corresponding steps, which is confirmed by the general theory. The corresponding Laurent expansions for the solutions will have the form

$$x(t) = t^{-2} \sum_{k=0}^{\infty} x_k t^k$$

for Eq. (1.79) and the form

$$x(t) = t^{-1} \sum_{k=0}^{\infty} x_k t^k$$

for Eq. (1.80).

By introducing the auxiliary variables $y = \dot{x}$, $z = x^{-1}$, the third and fourth Painlevé equations (1.81) and (1.82) are transformed into semi-quasihomogeneous systems of three equations:

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = y^2 z + e^t(ax^2 + b) + e^{2t}(cx^3 - dz), \quad \dot{z} = -yz^2, \\ &\dot{x} = y, \quad \dot{y} = \tfrac{1}{2}y^2 z + \tfrac{3}{2}x^3 + 4tx^2 + 2(t^2 - a)x + bz, \quad \dot{z} = -yz^2. \end{aligned}$$

The corresponding matrices **G** then have form $\mathrm{diag}(1, 2, -1)$, and the parameter β has the value -1. The truncated systems in the two cases are analogous. We can write them as

$$\dot{x} = y, \quad \dot{y} = Ay^2z + Bx^3, \quad \dot{z} = -yz^2, \tag{1.85}$$

where $A = 1$, $B = c$ for the third Painlevé equation, $A = 1/2$, $B = 3/2$ for the fourth.

System (1.85) has a particular solution in the form of the quasihomogeneous ray

$$x = \left(\frac{2-A}{B}\right)^{1/2} t^{-1}, \; y = -\left(\frac{2-A}{B}\right)^{1/2} t^{-2}, \; z = \left(\frac{2-A}{B}\right)^{1/2} t.$$

The eigenvalues of the Kovalevsky matrix that correspond to this solution equal $-1,\ 1,\ -2(A-2)$. Therefore, for the third and fourth Painlevé equation, they respectively equal $-1,\ 1,\ 2$ and $-1,\ 1,\ 3$.

Very complex computations, expedited with the aid of symbolic computation, show that logarithms don't appear at any of the corresponding stages, so that the desired solutions of Eqs. (1.81) and (1.82) can be represented by Laurent series of the form

$$x(t) = t^{-1} \sum_{k=0}^{\infty} x_k t^k,$$

consistent with their meromorphic nature.

However, it turns out that the fifth and sixth Painlevé equations (1.83) and (1.84) can't be represented as semi-quasihomogeneous systems without a supplementary "trick".

First consider the fifth Painlevé equation. In (1.83) we perform a logarithmic change of variable $\tau = \ln t$. Then, after the introduction of the auxiliary variables $y = x'$, $z = x^{-1}$ already described, where the prime denotes differentiation with respect to the new independent variable τ, (1.83) is written as a system of three equations:

$$\begin{aligned}
x' &= y,\\
y' &= y^2 \frac{z(z-3)}{2(z-1)} + (x-1)^2(ax+bz) + ce^{\tau}x + de^{2\tau}\frac{x(z+1)}{(z-1)},\\
z' &= -yz^2.
\end{aligned}$$

This system is semi-quasihomogeneous: $\mathbf{G} = \mathrm{diag}\,(\mathbf{1}, \mathbf{2}, -\mathbf{1})$, $\beta = -1$. The corresponding truncated system once again has the form (1.83) if, for the role of independent variable, we choose the logarithmic time τ. In this, $A = 3/2$, $B = a$, so that the Kovalevsky exponents are $-1,\ 1,\ 1$. Consequently logarithms can appear in the desired solution only in the first stage.

An analysis done with the aid of symbolic computation showed that this doesn't happen, so that the desired solution of the fifth Painlevé equation (1.83) can be expanded in a Laurent series in τ:

$$x(\tau) = \tau^{-1} \sum_{k=0}^{\infty} x_k \tau^k,$$

or, after reverting to the original independent variable t:

$$x(t) = \ln^{-1} t \sum_{k=0}^{\infty} x_k \ln^k t,$$

which shows that $t = 0$ is a logarithmic movable singularity.

In order to investigate the sixth Painlevé equation (1.84), we perform a logarithmic change of time with two singularities: $\tau = \ln(t(t-1))$. We further introduce the auxiliary variables $y = x'$, $z = x^{-1}$, where the prime denotes differentiation with respect to the new "time" τ. Equation (1.84) then takes the form of the first order system

$$\begin{aligned}
x' &= y,\\
y' &= y^2 \frac{z(\phi z^2 - 2z(\phi+1)+3)}{2(z-1)(\phi z-1)} + \frac{e^\tau y}{2\phi-1}\left(\frac{2}{2\phi-1} - \frac{z}{\phi z-1}\right) + \\
&\quad + \frac{x(x-1)(x-\phi)}{(2\phi-1)^2}\left(a + b\phi z^2 + c\frac{z^2(\phi-1)}{(z-1)^2} + d\frac{z^2\phi(\phi-1)}{(\phi z-1)^2}\right),\\
z' &= -yz^2.
\end{aligned}$$

where, for convenience, we introduce the notation

$$\phi = \phi(\tau) = \frac{1}{2}\left(1 \pm \sqrt{1+4e^\tau}\right).$$

The system obtained is negative semi-quasihomogeneous and the matrix which gives the quasihomogeneous scale again has the form $\mathbf{G} = \operatorname{diag}(1, 2, -1)$, $\beta = -1$. The truncated system has the form (1.84), where we have taken the new time τ as the independent variable, and where $A = 3/2$, $B = 4a/5$. Just as with the fifth Painlevé equation, the Kovalevsky exponents equal $-1,\ 1, 1$, so that logarithms can appear only in the first step.

For this case too, with the aid of symbolic computation, a system of equations was obtained for finding the first coefficients, and it was discovered that logarithms don't appear in their solution. Consequently, the desired solution can be constructed in the form of an ordinary Taylor series

$$x(\tau) = \tau^{-1} \sum_{k=0}^{\infty} x_k \tau^k,$$

from which we obtain the expansion with respect to the original independent variable t:

$$x(t) = \ln^{-1}(t(t-1)) \sum_{k=0}^{\infty} x_k \ln^k(t(t-1)).$$

Consequently, $t = 0$ and $t = 1$ represent logarithmic movable singularities. In order to investigate the character of the singularities of the solutions at infinity, we make the substitution $t \mapsto \frac{1}{t}$.

Example 1.4.7. Finally, we discuss V.V. Ten's hypothesis concerning the stability of isolated equilibrium states of dynamical systems with invariant measure in a space of odd dimension. Let

$$\dot{x} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n, \tag{1.86}$$

be an autonomous system of differential equations admitting an invariant measure with smooth density:

$$\operatorname{div}(\rho\mathbf{f}) = 0. \tag{1.87}$$

Let $\mathbf{x} = \mathbf{0}$ be an equilibrium position: $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. V.V. Ten proposed that if n is odd and if the equilibrium position $\mathbf{x} = \mathbf{0}$ is isolated, then it is Lyapunov stable. This hypothesis has important consequences: all isolated equilibrium points in a stationary fluid flow in three-dimensional Euclidean space would then be stable.

In a typical situation, system (1.86) has the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + O(|\mathbf{x}|), \quad \det \mathbf{A} \neq 0. \tag{1.88}$$

We rewrite the "continuity equation" (1.87):

$$\dot{\nu} = -\operatorname{div}\mathbf{f}, \quad \nu = \ln\rho.$$

Setting $\mathbf{x} = \mathbf{0}$ in this equation, we obtain the equality $\operatorname{tr} A = 0$. Consequently, the sum of all eigenvalues of the matrix $\mathbf{A}$ is equal to zero. Then at least one of the eigenvalues lies in the right half-plane. In fact, in the opposite case the spectrum of $\mathbf{A}$ would be distributed over the imaginary axis. But the sum of all its eigenvalues is zero and n is odd, so zero must then be an eigenvalue (since for a real matrix the eigenvalues come in conjugate pairs). On the other hand, this contradicts the assumption of a nonsingular matrix. Consequently, by Lyapunov's theorem, $\mathbf{x} = \mathbf{0}$ is a stable equilibrium position for the system (1.88).

It can be proved analogously that nondegenerate periodic trajectories of a dynamical system with invariant measure in an *even-dimensional* space are always stable. Recall that a periodic orbit is called *nondegenerate* if its multipliers are distinct from 1. This observation is also correct for nondegenerate *reducible* invariant tori of odd codimension, filled with conditionally periodic trajectories.

As proved in [119], assertions about stability may be extended to positive semi-quasihomogeneous systems of differential equations:

$$\mathbf{f}_m + \sum_{\alpha > m} \mathbf{f}_\alpha,$$

where $\mathbf{f}_k$ is a quasihomogeneous field of degree k with one and the same quasihomogeneity matrix $\mathbf{G}$. The only additional condition is that $\mathbf{x} = \mathbf{0}$ is the *unique* equilibrium point of the quasihomogeneous field $\mathbf{f}_m$.

In fact, inasmuch as n is odd and $\mathbf{x} = \mathbf{0}$ is the only zero of the quasihomogeneous field $\mathbf{f}_m$, there is (in agreement with Lemma 1.1.1) a nonzero vector $\mathbf{z}$ satisfying one of the equations

$$\mathbf{f}_m(\mathbf{z}) = -\mathbf{G}\mathbf{z} \quad \text{or} \quad \mathbf{f}_m(\mathbf{z}) = \mathbf{G}\mathbf{z}.$$

By Theorem 1.1.2, Eq. (1.86) in this case admits a solution with the asymptotic

$$Zt^{-G} \quad \text{or} \quad Z(-t)^{-G},$$

respectively, as $t \to +\infty$ or $t \to -\infty$. This solution tends to the equilibrium point $\mathbf{x} = \mathbf{0}$ as $t \to \pm\infty$. If there is a solution of the second type ("departing from the point $\mathbf{x} = \mathbf{0}$"), then obviously the equilibrium point is unstable. It remains to consider the case where there is a solution asymptotic to the equilibrium point as $t \to +\infty$.

We make use of an assertion that is interesting in itself.

Lemma 1.4.1. *Suppose that system (1.86) has an invariant measure and admits a nontrivial solution $t \mapsto x(t)$ that tends toward zero as $t \to +\infty$. Then the equilibrium point $\mathbf{x} = \mathbf{0}$ is unstable.*

In fact, let the point $x_0 = x(0)$ lie in some ε_0-neighborhood of zero. For arbitrary $\varepsilon > 0$ there is a small neighborhood U_ε of the point x_0 which, under the action of the phase flow over a certain time, appears in its entirety in an ε-neighborhood of the point $\mathbf{x} = \mathbf{0}$. However, by the ergodic theorem of Schwarzschild-Littlewood [173], almost all points distributed initially over U_ϵ leave the ϵ_0-neighborhood of zero. This proves the instability of the equilibrium point, inasmuch as each exiting trajectory intersects the ε-neighborhood of zero.

In the article [119] it is proved that Ten's hypothesis is not true, even for an *infinitely smooth* vector field $\mathbf{f}$, without some supplementary assumption about nonsingularity: in the construction of the counterexample there is a Maclaurin series of this vector field that is zero. But it is not impossible that Ten's hypothesis will turn out to be correct in the *analytic* case.

1.5 Group Theoretical Interpretation

Before beginning the exposition of new material, we return to the content of the first section. Our goal there was the construction of particular solutions of certain systems of differential equations with the help of particular solutions of a so-called truncated or model system. The choice of a truncation is dictated by some "scale" generated by the quasihomogeneous group of transformations (1.11) on extended phase space. The truncated system obtained is invariant under the given transformation group, and the "supporting" solution used lies on some orbit of this group. For the system of differential equations—whose right sides are represented as a series (1.4)—the quasihomogeneous scale introduced is rather natural. The question however arises about the possibility of using this or that one-parameter group of transformations of extended phase space for constructing such a scale. Another rather important problem consists of successively appending particular solutions to the truncated system obtained in this way until there is solution of the full system. It is to these questions that we devote the present section.

The material set forth below contains several facts from the theory of symmetry groups of ordinary differential equations. In order to obtain a fuller acquaintance with this material we recommend the monographs [145, 146].

We consider a smooth autonomous system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1.89}$$

and some one-parameter semigroup $\mathbf{X}$ of transformations from Ω to itself:

$$\mathbf{X}\colon \Omega[0, \chi \times \infty) \to \Omega,\ \ \chi = \pm 1,\ \ \mathbf{X}(\cdot, 0) = \mathrm{id}.$$

Let the smooth vector field $\mathbf{g}(\mathbf{x})$ be the infinitesimal generator, and let σ be the parameter of this semigroup. The parameter σ belongs either to a right or left half-line depending on the sign. We consider the system of differential equations

$$\frac{d\mathbf{x}}{d\sigma} = \mathbf{g}(\mathbf{x}). \tag{1.90}$$

that generates the given semigroup.

In general, we restrict the domain of variation of the parameter σ to some small finite interval $(-\sigma_0, +\sigma_0)$; then, according to the theorem on existence and uniqueness of solutions to the system (1.90), $\mathbf{X}$ will be a group. However, in the sequel we will likewise be interested in infinite values of the parameter σ, so that here it is more logical to speak about *semigroups* of transformations, inasmuch as it may happen that various elements of $\mathbf{X}$ will not have inverses (we suppose a priori that solutions exist, at least on half-lines). For the simplicity of the presentation we will henceforth ignore this fact and speak about groups of transformations. Likewise for simplicity, we will assume that the group considered acts globally on the domain Ω.

Definition 1.5.1. We say that $\mathbf{X}$ is a *generalized symmetry group* of the system (1.89) if, after the substitution

$$\mathbf{x} = \mathbf{X}(\mathbf{y}, \boldsymbol{\sigma}), \tag{1.91}$$

the system (1.89) assumes the form

$$\dot{\mathbf{y}} = \boldsymbol{\phi}(\mathbf{y}, \boldsymbol{\sigma})\mathbf{f}(\mathbf{y}), \tag{1.92}$$

where $\phi\colon \Omega \times [0, \chi \times \infty) \to R^+$ is some positive function. If $\mathbf{X} = \mathrm{id}$ for $\sigma = 0$, then $\phi(\mathbf{y}, \mathbf{0}) = \mathbf{1}$. If $\phi(\mathbf{y}, \boldsymbol{\sigma}) \equiv \mathbf{1}$ for arbitrary $\mathbf{y}$, $\boldsymbol{\sigma}$, then we will say that $\mathbf{X}$ is the *symmetry group* of (1.89).

If $\mathbf{X}$ is a symmetry group, then the corresponding transformations take solutions of system (1.89) to solutions of the very same system. But if $\mathbf{X}$ is an extended symmetry group, then $\mathbf{X}$ only leaves a family of trajectories invariant, and the parameterization of solutions changes [145, 146]. We prove one auxiliary result.

Lemma 1.5.1. *Let* $\mathbf{X}$ *be the extended symmetry group of system (1.89) that is generated by system (1.92). Then for arbitrary* $\in \mathbf{\Omega}$ *the following equation holds:*

$$[\mathbf{f}, \mathbf{g}](\mathbf{x}) = -\frac{\partial \phi}{\partial \sigma}(\mathbf{x}, 0)\mathbf{f}(\mathbf{x}), \tag{1.93}$$

where

$$[\mathbf{f}, \mathbf{g}] = (d\,\mathbf{g})\mathbf{f} - (d\,\mathbf{f})\mathbf{g}$$

and where the brackets are those for the Lie algebra of a smooth vector field.

Proof. We apply transformation (1.91) to system (1.89). Since

$$\dot{\mathbf{x}} = d_y \mathbf{X}(\mathbf{y}, \sigma)\dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}),$$

it follows from (1.92) that

$$\phi(\mathbf{y}, \sigma) d_y \mathbf{X}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}) \tag{1.94}$$

(for brevity, here and in the sequel we write $\mathbf{x}$ instead of $\mathbf{X}(\mathbf{y},\boldsymbol{\sigma})$).

In the sequel we will need the following assertion from the general theory of differential equations (see e.g. [42]).

Lemma 1.5.2. *Let* $\mathbf{U}(\mathbf{y},\boldsymbol{\sigma})$ *be the fundamental matrix of the linear system of differential equations*

$$\frac{d\mathbf{u}}{d\sigma} = d\,\mathbf{g}(\mathbf{x})\mathbf{u},$$

normalized by the condition $\mathbf{U}(\mathbf{y}, 0) = \mathbf{E}$. *Then the following identity holds:*

$$d_y \mathbf{X}(\mathbf{y}, \sigma) = \mathbf{U}(\mathbf{y}, \sigma). \tag{1.95}$$

We continue the proof of Lemma 1.5.1. We substitute (1.95) into (1.94) and differentiate with respect to σ. As a result we obtain the following identity:

$$\phi(\mathbf{y}, \sigma) d\,\mathbf{g}(\mathbf{x})\mathbf{U}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{y}) + \frac{\partial \phi}{\partial \sigma}(\mathbf{y}, \sigma)\mathbf{U}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{y}) = d\,\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x}).$$

Since $\mathbf{x} \to \mathbf{y}$ as $\sigma \to 0$, passing to the limit we get

$$d\,\mathbf{g}(\mathbf{y})\mathbf{f}(\mathbf{y}) + \frac{\partial \phi}{\partial \sigma}(\mathbf{y}, 0)\mathbf{f}(\mathbf{y}) = d\,\mathbf{f}(\mathbf{y})\mathbf{g}(\mathbf{y}).$$

Since, for fixed σ, the transformation $\mathbf{x}$ establishes a diffeomorphism of the domain Ω, returning to the variable $\mathbf{x}$ we obtain the relation (1.93).

The lemma is proved.

The converse also holds:

Lemma 1.5.3. *Suppose there exists a smooth function $\psi\colon \Omega \to R$ such that the identity*

$$[\mathbf{f}, \mathbf{g}](\mathbf{x}) = -\psi(\mathbf{x})\mathbf{f}(\mathbf{x}) \tag{1.96}$$

holds. Then $\mathbf{X}$ *(the phase flow of the vector field* $\mathbf{g}$*) is the extended symmetry group of system (1.89).*

Suppose that, under the action of the group $\mathbf{X}$, system (1.89) assumes the form

$$\dot{\mathbf{y}} = \phi(\mathbf{y}, \sigma)\widehat{\mathbf{f}}(\mathbf{y}, \sigma).$$

If we set

$$\phi(\mathbf{y}, \sigma) = \int_0^\sigma \psi(\mathbf{x})d\sigma,$$

then

$$\widehat{\mathbf{f}}(\mathbf{y}, \sigma) = \left(-\int_0^\sigma \psi(\mathbf{x})d\sigma\right)\mathbf{U}^{-1}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{x}). \tag{1.97}$$

We have to show that $\widehat{\mathbf{f}}(\mathbf{y}, \sigma) = \mathbf{f}(\mathbf{y})$ holds for arbitrary σ.

It follows at once from (1.97) that $\widehat{\mathbf{f}}(\mathbf{y}, 0) = \mathbf{f}(\mathbf{y})$. Differentiating (1.97) with respect to σ and using (1.96) and the formula for the differentiated inverse matrix, we obtain

$$\frac{\partial \widehat{\mathbf{f}}}{\partial \sigma}(\mathbf{y}, \sigma) = \exp\left(-\int_0^\sigma\right)\mathbf{U}^{-1}(\mathbf{y}, \sigma)(-d\mathbf{g}(x)\mathbf{f}(\mathbf{x}) - \\ -\psi(\mathbf{x})\mathbf{f}(\mathbf{x}) + d\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})) = 0,$$

from which the required assertion follows.

The lemma is proved.

We mention yet another technical result.

Lemma 1.5.4. *Let* $\mathbf{X}$ *be the extended symmetry group of system (1.89). After the change of variables*

$$\mathbf{x} = \mathbf{X}(\mathbf{y}, t) \tag{1.98}$$

system (1.89) assumes the form

$$\dot{\mathbf{y}} = \phi(\mathbf{y}, t)\mathbf{f}(\mathbf{y}) - \mathbf{g}(\mathbf{y}).$$

If $\mathbf{X}$ *is the symmetry group, then (1.89) can be rewritten in the form*

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) - \mathbf{g}(\mathbf{y}). \tag{1.99}$$

Proof. For brevity of notation we will, as before, set $\mathbf{x} = \mathbf{X}(\mathbf{y}, t)$. Then the identity

$$\dot{\mathbf{x}} = d_y \mathbf{X}(\mathbf{y}, t)\dot{\mathbf{y}} + \frac{\partial \mathbf{X}}{\partial t}(\mathbf{y}, t) = \mathbf{f}(\mathbf{x}).$$

holds.

Suppose that after the transformation (1.98), system (1.89) assumes the form

$$\dot{\mathbf{y}} = \mathbf{U}^{-1}(\mathbf{y}, t)\,(\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})) = \phi(\mathbf{y}, t)\left(\mathbf{f}(\mathbf{y}) - \tilde{\mathbf{f}}(\mathbf{y}, t)\right).$$

It is clear that for $t = 0$, we have $\tilde{\mathbf{f}}(\mathbf{y}, 0) = \mathbf{g}(\mathbf{y})$. We will compute the time derivative of $\tilde{\mathbf{f}}(\mathbf{y}, t)$. Using the formula for the differentiated inverse matrix, we obtain

$$\frac{\partial \tilde{\mathbf{f}}}{\partial t}(\mathbf{y}, t) = -\mathbf{U}^{-1}(\mathbf{y}, t)\,(d\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x}) - d\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})) = 0.$$

The lemma is proved.

Let the symmetry group of system (1.89) generate some solution of this system, i.e. suppose that there exists a $\mathbf{y}_0 \in \Omega$ such that $\mathbf{X}(\mathbf{y}_0, t)$ is a particular solution of (1.89). Then, for the autonomous system (1.99), $\mathbf{y} = \mathbf{y}_0$ will be a critical point, i.e.

$$\mathbf{f}_q(\mathbf{y}_0) = \mathbf{g}(\mathbf{y}_0). \tag{1.100}$$

In order to preserve the unity of the presentation, vector fields having group symmetry will be denoted $\mathbf{f}_{\mathrm{q}}$. As before, we can observe the analog for the Kovalevsky matrix

$$\mathbf{K} = d\mathbf{f}_q(\mathbf{y}_0) - d\mathbf{g}(\mathbf{y}_0). \tag{1.101}$$

Lemma 1.5.5. *Zero is always an eigenvalue of the matrix (1.101).*

Proof. Consider the vector $\mathbf{p} = \mathbf{f}_q(\mathbf{y}_0)$. Using the identity $[\mathbf{f}_q, \mathbf{g}] \equiv 0$ and the equality (1.100), we get

$$\mathbf{K}\mathbf{p} - d\mathbf{f}_q(\mathbf{y}_0)\mathbf{f}_q(\mathbf{y}_0) - d\mathbf{g}(\mathbf{y}_0)\mathbf{f}_q(\mathbf{y}_0) = d\mathbf{f}_q(\mathbf{y}_0)\left(\mathbf{f}_q(\mathbf{y}_0) - \mathbf{g}(\mathbf{y}_0)\right) = 0.$$

The lemma is proved.

Definition 1.5.2. We say that $\mathbf{X}$ is an *exponentially-asymptotic symmetry group* of system (1.89) if the right side of (1.89) can be expanded in a sum

$$\mathbf{f}(\mathbf{x}) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x})$$

such that, after the substitution (1.91), system (1.89) takes on the form

$$\dot{\mathbf{y}} = \sum_{m=0}^{\infty} e^{-m\beta\sigma} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}), \tag{1.102}$$

where $\mathrm{sign}\beta = \chi$ is analogous to the sign of semi-quasihomogeneity.

It is clear that the vector functions must satisfy the relation

$$[\mathbf{f}_{q+\chi m}, \mathbf{g}](\mathbf{x}) = m\beta \mathbf{f}_{q+\chi m}(\mathbf{x}). \tag{1.103}$$

Formally, for $\sigma \to \chi\times\infty$ and the substitution of $\mathbf{y}$ for $\mathbf{x}$, the system of differential equations (1.102) transforms to the "truncated" system

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}).$$

It is clear that after the substitution (1.98), system (1.89) assumes the form

$$\dot{\mathbf{y}} = \mathbf{f}_q(\mathbf{y}) - \mathbf{g}(\mathbf{y}) + \sum_{m=1} e^{-m\beta t}\mathbf{f}_{q+\chi m}(\mathbf{y}). \tag{1.104}$$

If the truncated system has the particular solution $\mathbf{X}(\mathbf{y}_0, t)$, discussed above, then taking $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$, we rewrite the system (1.104) in the form used in the Lemmas 1.3.2, 1.3.3, and 1.3.5:

$$\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{y}, t), \tag{1.105}$$

where

$$\boldsymbol{\phi}(\mathbf{u}) = \mathbf{f}_q(\mathbf{y}_0 + \mathbf{u}) - \mathbf{g}(\mathbf{y}_0 + \mathbf{u}) - \mathbf{K}\mathbf{u} - \mathbf{f}_q(\mathbf{y}_0) + \mathbf{g}(\mathbf{y}_0),$$
$$\boldsymbol{\psi}(\mathbf{u}, t) = \textstyle\sum_{m=1} e^{-m\beta t}\mathbf{f}_{q+\chi m}(\mathbf{y}_0 + \mathbf{u}).$$

Using the method developed in the preceding sections, we can assert that (1.105) always has a smooth particular solution that can be represented in the form of an exponential series

$$\mathbf{u}(t) = \sum_{k=1}^{\infty} \mathbf{u}_k(t)e^{-k\beta t}, \tag{1.106}$$

where the $\mathbf{u}_k$ are polynomial vector functions, which may be taken as constants if the matrix $\mathbf{K}$ doesn't have eigenvalues of the form $-k\beta$, $k \in \mathbb{N}$.

If all the coefficients $\mathbf{u}_k$ are constant, then in the analytic case, using the abstract implicit function theorem [94], it is possible to prove convergence for the series (1.106).

But if, in addition, the matrix $\mathbf{K}$ has l eigenvalues whose real parts have signs that coincide with the sign $-\beta$, and if the sign of the remaining real parts is opposite, then (1.105) possesses an l–parameter family of particular solutions tending to $\mathbf{u} = 0$ as $t \to \chi \times \infty$.

Thus we have established the following result.

Theorem 1.5.1. *Let system (1.89) have an exponentially-asymptotic group of symmetries, generating a particular solution* $\mathbf{X}(\mathbf{y}_0, t)$ *of the truncated system for some* $y_0 \in \Omega$. *Then the full system (1.89) has a particular solution of the form*

$$\mathbf{x}(t) = \mathbf{X}(\mathbf{y}_0 + o(1), t) \quad as \quad t \to \chi \times \infty.$$

Moreover, there exists an l–parameter family of such solutions, provided that the characteristic equation

$$\det(\mathbf{K} - \rho \mathbf{E}) = 0$$

has l roots for which the signs of the real parts coincide with the sign of $-\beta$, the real part of each of the remaining roots being zero or having the opposite sign.

Example 1.5.1. The classical Lyapunov first method can be given a group theoretic interpretation. Indeed, consider an autonomous system of differential equations for which $\mathbf{x} = \mathbf{0}$ is a critical point:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{m=1}^{\infty} \mathbf{f}_{m+1}(\mathbf{x}), \tag{1.107}$$

where $\mathbf{A}$ is a real matrix and $\mathbf{f}_{m+1}$ is a homogeneous vector function of degree $m + 1$.

As our first example, we consider the simplest system of linear differential equations

$$\frac{d\mathbf{x}}{d\sigma} = -\beta \mathbf{x}, \tag{1.108}$$

where β is some real number.

It is easily seen that system (1.108) generates a homogeneous group of dilations of the phase space $\mathbb{R}^n$, being the group of symmetries of the truncated linear system ($\mathbf{f}_q(\mathbf{x}) = \mathbf{A}\mathbf{x}$); and, for the full system, the designated group will be an exponentially–asymptotic symmetry group. If the number $-\beta$ is an eigenvalue of the matrix $\mathbf{A}$, then the truncated system will have a particular solution of the form $\mathbf{x} = e^{-\beta t}\mathbf{y}_0$ ($\mathbf{y}_0$ is a nonzero eigenvector of the matrix $\mathbf{A}$), situated on an orbit of this group. From Theorem 1.5.1 it then follows that system (1.107) has a particular solution in the form of the series

$$\mathbf{x}(t) = e^{-\beta t} \sum_{k=0}^{\infty} \mathbf{x}_k(t) e^{-k\beta t},$$

where $\mathbf{x}_k(t)$ is some polynomial function of the time t and $\mathbf{x}_0 \equiv \mathbf{y}_0$.

The existence of a family of solutions, converging to the exponential "supporting" solution $\mathbf{x} = e^{-\beta t}\mathbf{y}_0$ as $t \to \chi \times \infty$, depends on the number of eigenvalues of the matrix $\mathbf{K} = \mathbf{A} + \beta \mathbf{E}$ with positive (or negative) real part.

The method considered likewise makes it possible to construct solutions with *exponential asymptotic*, whose presence cannot be predicted by the *classical theory*.

Example 1.5.2. We consider the system of two differential equations

$$\dot{x} = 2x + ax^2 y, \quad \dot{y} = -3x^3 y^3 + bxy^2. \tag{1.109}$$

The phase flow of the linear system

$$\frac{dx}{d\sigma} = 2x, \quad \frac{dy}{d\sigma} = -3y$$

generates an exponentially-asymptotic symmetry group of (1.109), with $\beta = 1$.

In fact, introducing the notations

$$\mathbf{g} = (2x, -3y), \quad \mathbf{f}_{\mathrm{q}} = (2x, -3x^3y^3), \quad \mathbf{f}_{\mathrm{q}+1} = (ax^2y, bxy^2),$$

we obtain commutation relations of the type (1.103):

$$[\mathbf{f}_{\mathrm{q}}, \mathbf{g}] = (0, 0), \quad [\mathbf{f}_{\mathrm{q}+1}, \mathbf{g}] = \mathbf{f}_{\mathrm{q}+1}.$$

The truncated system

$$\dot{x} = 2x, \quad \dot{y} = -3x^3y^3$$

has a one-parameter family of particular solutions

$$x_0(c, t) = ce^{2t}, \quad y_0(c, t) = c^{-3/2}e^{-3t}, \quad c \in \mathbb{R}^+.$$

It is easy to compute the set of eigenvalues of the matrix $\mathbf{K}$ corresponding to the given family of solutions: $0, -12$.

Using Theorem 1.5.1 we obtain that the system of equations (1.109) has a one-parameter family of particular solutions, each represented as a series

$$x(c, t) = e^{2t} \sum_{k=0}^{\infty} x_k(t)e^{-kt}, \quad y(c, t) = e^{-3t} \sum_{k=0}^{\infty} y_k(t)e^{-kt},$$

where $x_0 = c$, $y_0 = c^{-3/2}$, and where nonconstant polynomials in t can appear only at the twelfth stage.

It is noteworthy that, as $t \to +\infty$, the solutions from the family constructed behave just like solutions of a linear hyperbolic system with characteristic exponents $2, -3$, although the linearized system only predicts the existence of solutions with asymptotic $\sim e^{2t}$ as $t \to -\infty$. The form of these solutions is obvious: the line $y = 0$ is a stable invariant manifold. On this solution manifold the solutions of (1.109) have the form

$$c(c_*, t) = c_*e^{2t}, \quad y(c_*, t) = 0.$$

Here c_* is the parameter for the given family of solutions.

Until now we have considered only the autonomous case and transformation groups whose action was applied to systems of differential equations under study and did not affect the independent variable.

We now consider a nonautonomous system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1.110}$$

that is given on some domain Ω of the extended phase space $\mathbb{R}^{n+1}$.

We can apply the method developed above to the nonautonomous system (1.110), represented as an autonomous system in extended phase space:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \varphi), \quad \dot{\varphi} = 1.$$

But the nonautonomous case has its very own peculiarities. In order to indicate them, we again construct the necessary theory.

Let some one-parameter transformation semigroup act on Ω:

$$(\mathbf{X}, T)\colon \Omega \times [0, \chi \times \infty) \to \Omega, \quad \chi = \pm 1, \quad (\mathbf{X}, T)(\cdot, \cdot, 0) = \mathrm{id},$$

yielding the system of differential equations

$$\frac{d\mathbf{x}}{d\sigma} = \mathbf{g}(\mathbf{x}, t), \quad \frac{dt}{d\sigma} = h(\mathbf{x}, t). \tag{1.111}$$

It is further assumed that the smooth function $h(\mathbf{x}, t)$ doesn't vanish on the domain Ω.

We consider a change of dependent and independent variables generated by the system of differential equations (1.111)

$$\mathbf{x} = \mathbf{X}(\mathbf{y}, \tau, \sigma), \quad t = T(\mathbf{y}, \tau, \sigma), \quad \sigma = \text{const}. \tag{1.112}$$

Suppose that, under the action of the substitution (1.112), system (1.110) assumes the form

$$\mathbf{y}' = \widehat{\mathbf{f}}(\mathbf{y}, \tau, \sigma), \tag{1.113}$$

where the prime denotes differentiation with respect to the new independent variable τ. It is clear that $\widehat{\mathbf{f}}(\mathbf{y}, \tau, 0) = \mathbf{f}(\mathbf{y}, \tau)$.

Lemma 1.5.6. *In order that the system (1.110) be transformed by the action of the substitution (1.113) into the form (1.113), it is necessary and sufficient that, for arbitrary $(\mathbf{x}, t) \in \Omega$, the following equation be satisfied:*

$$(L^{\mathbf{g}}\mathbf{f})(\mathbf{x}, t) + \mathbf{d}(\mathbf{x}, t) - h(\mathbf{x}, t)\frac{\partial \mathbf{f}}{\partial t}(\mathbf{x}, t) - (D_t^{\mathbf{f}} h)(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t) = 0, \tag{1.114}$$

where $L^{\mathbf{g}} = \frac{\partial \mathbf{g}}{\partial t} + [\cdot, \mathbf{g}]$ is the Lie operator, $D_t^{\mathbf{f}} = \frac{\partial}{\partial t} + \langle d_{\mathbf{x}}, \mathbf{f} \rangle$ is the total derivative operator with respect to time by virtue of the system of equations (1.110), and where $\mathbf{d}\colon \Omega \to \mathbb{R}^n$ is a smooth vector function satisfying

$$\mathbf{d}(\mathbf{x}, t) = \frac{\partial \widehat{\mathbf{f}}}{\partial \sigma}(\mathbf{x}, t, 0). \tag{1.115}$$

Proof. We first compute the differentials of the left and right sides of (1.112):

$$\begin{aligned} d\mathbf{x} &= d_y\mathbf{X}(\mathbf{y},\tau,\sigma)\,d\mathbf{y} + \tfrac{\partial\mathbf{X}}{\partial\tau}(\mathbf{y},\tau,\sigma)\,d\tau + \tfrac{\partial\mathbf{X}}{\partial\sigma}(\mathbf{y},\tau,\sigma)\,d\sigma,\\ dt &= \langle d_yT(\mathbf{y},\tau,\sigma),\,d\mathbf{y}\tfrac{\partial T}{\partial\tau}(\mathbf{y},\tau,\sigma)\,d\tau + \tfrac{\partial T}{\partial\sigma}(\mathbf{y},\tau,\sigma)\,d\sigma. \end{aligned} \tag{1.116}$$

Since the parameter σ is fixed and the differential relations $d\mathbf{x} = \mathbf{f}(\mathbf{x},t)\,dt$, $d\mathbf{y} = \widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma)\,d\tau$ hold, from (1.116) we obtain the equation

$$\begin{aligned} &d_\mathbf{y}\mathbf{X}(\mathbf{y},\tau,\sigma)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) + \tfrac{\partial\mathbf{X}}{\partial\sigma}(\mathbf{y},\tau,\sigma) =\\ &\quad= \left\{\left\langle d_\mathbf{y}T(\mathbf{y},\tau,\sigma),\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma)\right\rangle + \tfrac{\partial T}{\partial\tau}(\mathbf{y},\tau,\sigma)\right\}\mathbf{f}(\mathbf{x},t), \end{aligned} \tag{1.117}$$

where (as earlier) the relations $\mathbf{x} = \mathbf{X}(\mathbf{y},\tau,\sigma)$, $t = T(\mathbf{y},\tau,\sigma)$ are satisfied.

In the remainder of this section, for notational brevity in multiplying vector objects of various dimensions, we will use the standard notation of matrix algebra. Here vector fields and vector functions $\mathbf{f},\mathbf{g},\mathbf{X}$, and their derivatives with respect to the "time-like" variables t,τ,σ, are vectors (i.e. $n\times 1$ matrices), and the differentials of the scalar functions h, T with respect to the spatial variables $\mathbf{x},\mathbf{y}$ are covectors (i.e. $1\times n$ matrices). All scalar quantities are considered to be 1×1 matrices.

According to Lemma 1.5.2, the matrix $d_\mathbf{y}\mathbf{X}(\mathbf{y},\tau,\sigma)$, the vector $\frac{\partial\mathbf{X}}{\partial\tau}(\mathbf{y},\tau,\sigma)$, the covector $d_\mathbf{y}T(\mathbf{y},\tau,\sigma)$ and the function $\frac{\partial T}{\partial\tau}(\mathbf{y},\tau,\sigma)$ satisfy the matrix differential equation

$$\frac{\partial}{\partial\sigma}\begin{pmatrix} d_\mathbf{y}\mathbf{X} & \frac{\partial}{\partial\tau}\mathbf{X}\\ d_\mathbf{y}T & \frac{\partial}{\partial\tau}T\end{pmatrix} = \begin{pmatrix} d_\mathbf{x}\mathbf{g}(\mathbf{x},t) & -\frac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t)\\ d_\mathbf{x}h(\mathbf{x},t) & \frac{\partial}{\partial t}h(\mathbf{x},t)\end{pmatrix}\begin{pmatrix} d_\mathbf{y}\mathbf{X} & \frac{\partial}{\partial\tau}\mathbf{X}\\ d_\mathbf{y}T & \frac{\partial}{\partial\tau}T\end{pmatrix} \tag{1.118}$$

and the initial conditions

$$\begin{aligned} d_\mathbf{y}\mathbf{X}(\mathbf{y},\tau,0) &= \mathbf{E}, & \tfrac{\partial\mathbf{X}}{\partial\tau}(\mathbf{y},\tau,0) &= 0,\\ d_\mathbf{y}T(\mathbf{y},\tau,0) &= 0, & \tfrac{\partial T}{\partial\tau}(\mathbf{y},\tau,0) &= 1. \end{aligned} \tag{1.119}$$

Differentiating relation (1.117) with respect to the variable σ, using (1.118) and (1.119), passing to the limit as $\sigma\to 0$ and letting $\mathbf{x}\to\mathbf{y}$, $t\to\tau$, we obtain

$$\begin{aligned} d_\mathbf{y}\mathbf{g}(\mathbf{y},\tau)\mathbf{f}(\mathbf{y},\tau) + \tfrac{\partial\widehat{\mathbf{f}}}{\partial\sigma}(\mathbf{y},\tau,0) + \tfrac{\partial\mathbf{g}}{\partial\tau}(\mathbf{y},\tau) =\\ = \mathbf{f}(\mathbf{y},\tau)\left\{d_\mathbf{y}h(\mathbf{y},\tau)\mathbf{f}(\mathbf{y},\tau) + \tfrac{\partial h}{\partial\tau}(\mathbf{y},\tau)\right\} +\\ + d_\mathbf{y}\mathbf{f}(\mathbf{y},\tau)\mathbf{g}(\mathbf{y},\tau) + h(\mathbf{y},\tau)\tfrac{\partial\mathbf{f}}{\partial\tau}(\mathbf{y},\tau). \end{aligned}$$

Returning to the original phase variables $\mathbf{x}, t$ and introducing the notation (1.115), we get Eq. (1.114). Necessity is proved.

Now suppose that Eq. (1.114) is satisfied. We then show that the substitution (1.112) transforms system (1.110) into the form (1.111), with fulfillment of (1.115).

We write (1.117) in the following form:

$$\mathbf{U}(\mathbf{y},\tau,\sigma)\widehat{\mathbf{f}}(\mathbf{y},\tau,0) = \mathbf{q}(\mathbf{y},\tau,\sigma),$$

with the notations

$$\mathbf{U}(\mathbf{y},\tau,\sigma) = d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma),$$
$$\mathbf{q}(\mathbf{y},\tau,\sigma) = -\tfrac{\partial\mathbf{X}}{\partial\tau}(\mathbf{y},\tau,\sigma) + \mathbf{f}(\mathbf{x},t)\tfrac{\partial T}{\partial\tau}(\mathbf{y},\tau,\sigma).$$

From (1.119) it follows that $\mathbf{U}(\mathbf{y},\tau,0) = \mathbf{E}$, whereby the matrix $\mathbf{U}(\mathbf{y},\tau,\sigma)$ is invertible for small values of σ. That is

$$\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\mathbf{q}(\mathbf{y},\tau,\sigma),$$

from which it follows that $\widehat{\mathbf{f}}(\mathbf{y},\tau,0) = \mathbf{f}(\mathbf{y},\tau)$.

We compute the derivative

$$\begin{aligned}\tfrac{\partial\widehat{\mathbf{f}}}{\partial\sigma}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\times \\ \times\left(\tfrac{\partial\mathbf{U}}{\partial\sigma}(\mathbf{y},\tau,\sigma)\mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\mathbf{q}(\mathbf{y},\tau,\sigma) - \tfrac{\partial\mathbf{q}}{\partial\sigma}(\mathbf{y},\tau,\sigma)\right).\end{aligned} \tag{1.120}$$

Then the following assertion holds:

Lemma 1.5.7. *The matrix* $\mathbf{U}(\mathbf{y},\tau,\sigma)$ *and the vector* $\mathbf{q}(\mathbf{y},\tau,\sigma)$ *satisfy the matrix nonhomogeneous differential equations*

$$\frac{\partial\mathbf{U}}{\partial\sigma}(\mathbf{y},\tau,\sigma) = \mathbf{A}(\mathbf{x},t)\mathbf{U}(\mathbf{y},\tau,\sigma) - \mathbf{d}(\mathbf{x},t)d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma), \tag{1.121}$$

$$\frac{\partial\mathbf{q}}{\partial\sigma}(\mathbf{y},\tau,\sigma) = \mathbf{A}(\mathbf{x},t)\mathbf{q}(\mathbf{y},\tau,\sigma) - \mathbf{d}(\mathbf{x},t)\frac{\partial T}{\partial\tau}(\mathbf{y},\tau,\sigma), \tag{1.122}$$

where

$$\mathbf{A}(\mathbf{x},t) = d_{\mathbf{x}}\mathbf{g}(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{x}}h(\mathbf{x},t).$$

Proof. Using the matrix differential equation (1.118), we obtain

$$\begin{aligned}\tfrac{\partial\mathbf{U}}{\partial\sigma}(\mathbf{y},\tau,\sigma) = (d_{\mathbf{x}}\mathbf{g}(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{x}}h(\mathbf{x},t))\,d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma)+ \\ +\left(\tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t) - d_{\mathbf{x}}\mathbf{f}(\mathbf{x},t)\mathbf{g}(\mathbf{x},t) - \tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t)h(\mathbf{x},t)- \right.\\ \left. -\mathbf{f}(\mathbf{x},t)\tfrac{\partial h}{\partial t}(\mathbf{x},t)\right)d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma).\end{aligned}$$

Using Eq. (1.114), we arrive at (1.121). Equation (1.122) is obtained analogously. In fact,

$$\begin{aligned}\tfrac{\partial\mathbf{q}}{\partial\sigma}(\mathbf{y},\tau,\sigma) = -\,(d_{\mathbf{x}}\mathbf{g}(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{x}}h(\mathbf{x},t))\,\tfrac{\partial}{\partial\tau}\mathbf{X}(\mathbf{y},\tau,\sigma)- \\ -\left(\tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t) - d_{\mathbf{x}}\mathbf{f}(\mathbf{x},t)\mathbf{g}(\mathbf{x},t) - \tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t)h(\mathbf{x},t)- \right.\\ \left. -\mathbf{f}(\mathbf{x},t)\tfrac{\partial h}{\partial t}(\mathbf{x},t)\right)\tfrac{\partial}{\partial\tau}T(\mathbf{y},\tau,\sigma).\end{aligned}$$

Using (1.114) once more, we obtain (1.122).

Lemma 1.5.7 is proved.

We continue the proof of Lemma 1.5.6. Substituting Eq. (1.121) and (1.122) into (1.120):

$$\frac{\partial\widehat{\mathbf{f}}}{\partial\sigma}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\times$$
$$\times\left(\mathbf{d}(\mathbf{x},t)d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma)\mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\mathbf{q}(\mathbf{y},\tau,\sigma) + \mathbf{d}(\mathbf{x},t)\tfrac{\partial T}{\partial\tau}(\mathbf{y},\tau,\sigma)\right).$$

Passing to the limit as $\sigma \to 0$, we obtain the relation (1.115).

The lemma is proved.

In the change of variables (1.112), the parameter σ was fixed. We now fix τ and consider the transformation $(\mathbf{x}, t) \mapsto (\mathbf{y}, \sigma)$ of extended phase space:

$$\mathbf{x} = \mathbf{X}(\mathbf{y},\tau,\sigma), \quad = T(\mathbf{y},\tau,\sigma), \quad \tau = \text{const.} \tag{1.123}$$

Lemma 1.5.8. *Under the action of transformation (1.123) system (1.57) assumes the form*

$$\frac{d\mathbf{y}}{d\sigma} = h(\mathbf{y},\tau)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) - \mathbf{g}(\mathbf{y},\tau), \tag{1.124}$$

where τ now takes on the role of parameter.

Proof. Let $d\mathbf{y} = \widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma)\,d\sigma$. Since τ is fixed, (1.116) yields the equation

$$\begin{aligned} &d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) + \tfrac{\partial\mathbf{X}}{\partial\sigma}(\mathbf{y},\tau,\sigma) = \\ &\qquad = \mathbf{f}(\mathbf{x},t)\left\{d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma)\widetilde{\mathbf{f}}(\mathbf{y},\tau,\sigma) + \tfrac{\partial T}{\partial\sigma}(\mathbf{y},\tau,\sigma)\right\}. \end{aligned} \tag{1.125}$$

Equation (1.125) can be rewritten in the form

$$\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\left(-\frac{\partial\mathbf{X}}{\partial\sigma}(\mathbf{y},\tau,\sigma) + \mathbf{f}(\mathbf{x},t)\frac{\partial T}{\partial\sigma}(\mathbf{y},\tau,\sigma)\right).$$

We use the local invertibility of the change of variables (1.112). The one-parameter (semi)group of transformations

$$\mathbf{y} = \mathbf{X}(\mathbf{x},\tau,-\sigma), \quad \tau = T(\mathbf{y},\tau,\sigma) \tag{1.126}$$

represents the phase flow of the system of differential equations

$$\frac{d\mathbf{y}}{d\sigma} = -\mathbf{g}(\mathbf{y},\tau), \quad \frac{d\tau}{d\sigma} = -h(\mathbf{y},\tau). \tag{1.127}$$

Substituting (1.126) into (1.112), we obtain an identity for an arbitrary triple $(\mathbf{x},\tau,\sigma) \in \Omega \times (-\sigma_0, +\sigma_0)$. Differentiating this identity with respect to σ and using (1.127), we obtain

$$0 = -d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma)\mathbf{g}(\mathbf{y},\tau) - \tfrac{\partial\mathbf{X}}{\partial\tau}(\mathbf{y},\tau,\sigma)h(\mathbf{y},\tau) + \tfrac{\partial\mathbf{X}}{\partial\sigma}(\mathbf{y},\tau,\sigma),$$
$$0 = -d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma)\mathbf{g}(\mathbf{y},\tau) - \tfrac{\partial T}{\partial\tau}(\mathbf{y},\tau,\sigma)h(\mathbf{y},\tau) + \tfrac{\partial\mathbf{X}}{\partial\sigma}(\mathbf{y},\tau,\sigma).$$

Then, since

$$\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\left(-\frac{\partial \mathbf{X}}{\partial \tau}(\mathbf{y},\tau,\sigma) + \mathbf{f}(\mathbf{x},\tau)\frac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma)\right),$$

we easily obtain

$$\widetilde{\mathbf{f}}(\mathbf{y},\tau,\sigma) = h(\mathbf{y},\tau)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) - \mathbf{g}(\mathbf{y},\tau).$$

The lemma is proved.

We now return to the study of symmetry groups for nonautonomous systems of differential equations.

Definition 1.5.3. We say that $(\mathbf{X}, T)$ is a *extended symmetry group* of system (1.110) if (1.110) is invariant under the action of the transformation (1.112).

The introduction of the additional term "extended" in the definition is necessary in order to distinguish symmetry groups of autonomous systems—acting on generalized phase space—from symmetry groups not affecting the independent variable.

From Lemma 1.5.6 it follows that, in order that $(\mathbf{X}, T)$ be an extended symmetry group for (1.110), it is necessary and sufficient that, for arbitrary $(\mathbf{x}, t) \in \Omega$, the following relation be satisfied:

$$\left(L^{\mathbf{g}}\mathbf{f}\right)(\mathbf{x},t) - h(\mathbf{x},t)\frac{\partial \mathbf{f}}{\partial t}(\mathbf{x},t) - \left(D^{\mathbf{f}}h\right)(\mathbf{x},t)\mathbf{f}(\mathbf{x},t) = 0. \tag{1.128}$$

If $(\mathbf{X}, T)$ is the extended symmetry group of system (1.110) then, after the substitution (1.123), this system assumes the form

$$\frac{d\mathbf{y}}{d\sigma} = h(\mathbf{y},\tau)\mathbf{f}(\mathbf{y},\tau) - \mathbf{g}(\mathbf{y},\tau), \tag{1.129}$$

where τ is here some fixed parameter, i.e. under the action of the transformation (1.123) the resulting system becomes autonomous.

Let the system considered have some particular solution, situated on an orbit of its extended symmetry group. This implies the existence of $(\mathbf{y}_0, \tau_0) \in \Omega$ such that the given parametric vector function $\mathbf{x} = \mathbf{X}(\mathbf{y}_0, \tau_0, \sigma)$, $t = (\mathbf{y}_0, \tau_0, \sigma)$ will be a particular solution of (1.110). Such a solution will be smooth provided that, as was stipulated above, the function $h(\mathbf{x}, t)$ is not transformed into zero. Then for $\tau = \tau_0$ the system of equations (1.129) has a critical point $\mathbf{y} = \mathbf{y}_0$, so that

$$h(\mathbf{y}_0,\tau_0)\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0,\tau_0) = \mathbf{g}(\mathbf{y}_0,\tau_0). \tag{1.130}$$

Further, as was done earlier, the right sides of systems of equations possessing extended symmetry groups will be assigned an index "q".

An analog of the Kovalevsky matrix will serve as the matrix of a linear system of differential equations with constant coefficients, obtained from (1.129) by linearization in the neighborhood of $\mathbf{y} = \mathbf{y}_0$:

$$\mathbf{K} = h(\mathbf{y}_0, \tau_0) d_{\mathbf{y}} \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0, \tau_0) + \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0, \tau_0) d_{\mathbf{y}} h(\mathbf{y}_0, \tau_0) - d_{\mathbf{y}} \mathbf{g}(\mathbf{y}_0, \tau_0). \tag{1.131}$$

We have the following

Lemma 1.5.9. *Let* $\frac{\partial \mathbf{f}_{\mathrm{q}}}{\partial t} \equiv 0$, $\frac{\partial \mathbf{g}}{\partial t} \equiv 0$. *Then the number* $-\frac{\partial h}{\partial \tau}(\mathbf{y}_0, \tau_0)$ *is an eigenvalue of the matrix (1.131).*

Proof. We consider the vector $\mathbf{p} = \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0)$. Using Eqs. (1.128) and (1.130), we compute

$$\mathbf{K}\mathbf{p} = d_{\mathbf{y}} \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) \left(h(\mathbf{y}_0, \tau_0) \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) - \mathbf{g}(\mathbf{y}_0) \right) - \\ - \tfrac{\partial h}{\partial \tau}(\mathbf{y}_0, \tau_0) \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) = - \tfrac{\partial h}{\partial \tau}(\mathbf{y}_0, \tau_0) \mathbf{p}.$$

The lemma is proved.

Definition 1.5.4. We say that $(\mathbf{X}, T)$ is an *exponentially-asymptotic generalized symmetry group* of system (1.110) if the right side of (1.110) can be expanded in a sum

$$\mathbf{f}(\mathbf{x}, t) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x}, t)$$

such that, after the substitution (1.112), system (1.110) assumes the form

$$\mathbf{y}' = \sum_{m=0}^{\infty} e^{-m\beta\sigma} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}, \tau). \tag{1.132}$$

If a system possesses an exponentially-asymptotic generalized symmetry group then, in accordance with Lemma 1.5.6, the vector function $\mathbf{f}(\mathbf{x}, t)$ satisfies a system of partial differential equations of type (1.114). Since the truncations $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ satisfy system (1.128), the system of equations for the perturbations can be written in the form

$$\sum_{m=1}^{\infty} \Big\{ h(\mathbf{x}, t) \frac{\partial \mathbf{f}_{\mathrm{q}+\chi m}}{\partial t}(\mathbf{x}, t) - \left(\mathbf{f}_{\mathrm{q}+\chi m}, \mathbf{g} \right)(\mathbf{x}, t) + \\ + \left(m\beta + \tfrac{\partial h}{\partial t}(\mathbf{x}, t) + \langle d_{\mathbf{x}} h, \mathbf{f} \rangle (\mathbf{x}, t) \right) \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x}, t) \Big\} = 0. \tag{1.133}$$

It is clear (as it was earlier) that, with convergence of the parameter σ to $\chi \times \infty$ and the formal substitution $(\mathbf{y}, \tau)$ into $(\mathbf{x}, t)$, system (1.132) goes over to the truncated system

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t).$$

The transformation of generalized phase space (1.123) reduces a system possessing an exponentially asymptotic generalized symmetry group to the form

$$\frac{d\mathbf{y}}{d\sigma} = h(\mathbf{y}, \tau) \sum_{m=0}^{\infty} e^{-m\beta\sigma} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}, \tau) - \mathbf{g}(\mathbf{y}, \tau). \tag{1.134}$$

But, if the truncated system has a particular solution that is situated on an orbit of the group considered, then (1.134), with the aid of the "perturbing" substitution $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$, may be rewritten in the customary form

$$\frac{d\mathbf{u}}{d\sigma} = \mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{u}, \sigma), \tag{1.135}$$

where

$$\boldsymbol{\phi}(\mathbf{u}) = h(\mathbf{y}_0 + \mathbf{u}, \tau_0)\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0 + \mathbf{u}, \tau_0) - \mathbf{g}(\mathbf{y}_0 + \mathbf{u}, \tau_0) - \\ -\mathbf{K}\mathbf{u} - h(\mathbf{y}_0, \tau_0)\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0, \tau_0) - \mathbf{g}(\mathbf{y}_0, \tau_0),$$

and

$$\boldsymbol{\psi}(\mathbf{u}, \sigma) = h(\mathbf{y}_0 + \mathbf{u}, \tau_0) \sum_{m=1}^{\infty} e^{-m\beta\sigma} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}_0 + \mathbf{u}, \tau_0).$$

We need to remember that the right side of (1.135) depends on τ_0 only as a parameter and that for this reason this dependency is not reflected in (1.135).

In this section we have already considered the asymptotic properties of solutions of system (1.135) since, within substitution of σ for t, it coincides with system (1.105). These properties by themselves imply the following assertion about the properties of particular solutions of the original system.

Theorem 1.5.2. *Let system (1.110) possess an exponentially-asymptotic generalized symmetry group, on whose orbit there is a particular solution of the truncated system, having in parametric form the aspect*

$$\mathbf{x} = \mathbf{X}(\mathbf{y}_0, \tau_0, \sigma), \quad t = T(\mathbf{y}_0, \tau_0, \sigma),$$

for some $(\mathbf{y}_0, \tau_0) \in \Omega$. *Then the full system (1.110) has a particular solution of the form*

$$\mathbf{x}(\sigma) = \mathbf{X}\left(\mathbf{y}_0 + \mathbf{u}(\sigma), \tau_0, \sigma\right), \quad t(\sigma) = T\left(\mathbf{y}_0 + \mathbf{u}(\sigma), \tau_0, \sigma\right),$$

where $\mathbf{u} = o(1)$ *as* $\sigma \to \chi \times \infty$.

We can furthermore assert that there exists an l–parameter family of such solutions if the characteristic equation

$$\det(\mathbf{K} - \rho\mathbf{E}) = 0$$

has l roots, the signs of whose real parts coincide with the sign of $-\beta$, and the real part of each of the remaining roots is either zero or has the opposite sign.

We consider an example that illustrates the theorem.

Example 1.5.3. Let a relativistic particle, whose rest mass equals unity, complete a straight line motion along the Ox axis under the action of the force $f(x, t)$. The equation of motion of this particle has the form [128]

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1 - c^{-2}\dot{x}^2}}\right) = f(x, t), \tag{1.136}$$

where c is the speed of light.

It is known that, in the absence of exterior force fields ($f(x,t) \equiv 0$), Eq. (1.136) is invariant under the one-dimensional subgroup of the Lorentz group, being a group of hyperbolic rotations. The orbits of this group are “pseudospherical” in the Minkowski metric $(ds)^2 = c^2(dt)^2 - (dx)^2$ and have the form

$$\begin{aligned} x &= y\cosh\sigma + c\tau\sinh\sigma, \\ t &= c^{-1}y\sinh\sigma + \tau\cosh\sigma, \quad \sigma \in (-\infty, +\infty). \end{aligned} \tag{1.137}$$

Using the extension of the group action operation [146], it is also not difficult to compute the law of the speed change:

$$\dot{x} = c\,\frac{y' + c\sigma}{y'\sigma + c}, \tag{1.138}$$

where the prime denotes differentiation with respect to the new “time” τ.

From (1.138) it is clear that, as $\sigma \to \pm\infty$, the speed of the particle tends toward the speed of light.

But, in the absence of exterior fields, the unique trajectory situated on the orbits of this group corresponds to the singular solution (1.136) $x = ct$. We first find a nontrivial force field $f(x,t)$ such that Eq. (1.110) will be invariant with respect to the designated subgroup of the Lorentz group and will possess particular (nonsingular) solutions lying on its orbit. This in fact would indicate the acceleration toward the speed of light as $t \to \infty$.

We introduce the notation $u = \dot{x}$, $v = y'$ and write a system of three first order differential equations for which formulas (1.137) and (1.138) would give the general solution:

$$\frac{dx}{d\sigma} = ct, \quad \frac{du}{d\sigma} = c^{-1}(c^2 - u^2), \quad \frac{dt}{d\sigma} = c^{-1}x.$$

We rewrite the second order Eq. (1.136) under investigation as an equivalent system of two first order equations

$$\dot{x} = u, \quad \dot{u} = (1 - c^{-2}u^2)^{3/2} f(x,t).$$

Introducing the notations

$$\begin{aligned} \mathbf{x} = (x, u), \quad \mathbf{f} = \left(u, (1 - c^{-2}u^2)^{3/2} f(x,t)\right), \\ \mathbf{g} = \left(ct, c^{-1}(c^2 - u^2)\right), \quad h = c^{-1}x, \end{aligned}$$

we attempt to solve the system of equations (1.128).

It is easily observed that the structure of the equations considered is such that

$$\left(D_t^{\mathbf{f}} h\right)(\mathbf{x}, t) = c^{-1}u$$

does not depend on the concrete form of $f(x,t)$, thanks to which Eq. (1.128) becomes linear.

This system consists of two equations, the first satisfying an identity and the second leading to a linear equation in the first order partial derivatives of the function $f(x,t)$:

$$c^2 t \frac{\partial f}{\partial x}(x,t) + x \frac{\partial f}{\partial t}(x,t) = 0,$$

which has the obvious solution

$$f_{\mathrm{q}}(x,t) = \phi(x^2 - c^2 t^2),$$

where ϕ is an arbitrary smooth function.

It is interesting to note that the function f_{q} we have found is an invariant under the group of hyperbolic rotations. This function is ascribed the index "q", indicating invariance of the Eq. (1.136) with respect to hyperbolic rotations.

We consider the problem of existence of point trajectories, situated on the orbits of the group considered, such that the speed of the point after finite time does *not* attain the speed of light. For this it is necessary to solve a system of algebraic equations of type (1.130). After a few calculations we arrive at the equation

$$\xi_0 \phi(\xi_0^2) = c^2,$$

where we have introduced the notation $\xi_0 = y_0 \sqrt{1 - c^{-2} v_0^2}$. This equation has real roots, at least for functions ϕ that are positive and nondecreasing on $[0, +\infty)$.

The initial moment of time τ_0 is determined here by the formula

$$\tau_0 = c^{-2} y_0 v_0.$$

So, as was emphasized earlier, the trajectories with the determined initial conditions accelerate to the speed of light as $t \to \infty$.

We now pose the following question: which additional perturbing forces allow the existence of trajectories with analogous asymptotic properties?

We expand the function f in the formal series

$$f(x,t) = \sum_{m=0} f_{\mathrm{q}+\chi m}(x,t),$$

such that, subsequent to the transformation (1.137), the terms of Eq. (1.136) that corresponding to terms of this series are multiplied by $e^{-m\beta\sigma}$.

In order to find the functions $f_{\mathrm{q}+\chi m}$, it is necessary to solve a system analogous to (1.133). Thanks to the rather simple structure of the original Eq. (1.136), the corresponding system becomes linear and can furthermore be decomposed into a countable number of independent subsystems. The left and right sides of the first equations are identically zero and the second is reduced to the form

$$c^2 t \frac{\partial}{\partial x} f_{\mathrm{q}+\chi m}(x,t) + x \frac{\partial}{\partial t} f_{\mathrm{q}+\chi m}(x,t) - m\beta c f_{\mathrm{q}+\chi m}(x,t) = 0.$$

It is easy to see that the solutions of this equation will be function of the form

$$f_{\mathrm{q}+\chi m}(x,t) = \psi(x^2 - c^2t^2)\left(a_+(x+ct)^{m\beta} + a_-(x-ct)^{-m\beta}\right),$$

where ψ is an arbitrary function and a_+, a_- are arbitrary constants.

Thus the equation of motion of a particle in the constructed force field will have a particular solution of the form

$$\begin{aligned} x(\sigma) &= (y_0 + z(\sigma))\cosh\sigma + c\tau_0 \sinh\sigma, \\ t(\sigma) &= c^{-1}(y_0 + z(\sigma))\sinh\sigma + \tau_0\cosh\sigma, \end{aligned}$$

where the function $z(\sigma)$ can be expanded in a series

$$z(\sigma) = \sum_{k=1}^{\infty} z_k(\sigma)e^{-k\beta\sigma}.$$

This solution clearly possesses the very same asymptotic properties as the corresponding particular solution of the truncated system.

We note that the problem just solved represents the simplest example of an inverse problem in relativistic mechanics.

The systems of Eqs. (1.128) and (1.133) that must be satisfied by the right sides of systems of ordinary equations possessing a given generalized symmetry group, or exponentially-asymptotic generalized symmetry group in the nonautonomous case, are substantially more complicated than the corresponding systems (1.103) obtained for the autonomous case, in part because their nonlinearity makes it impossible in general to obtain a chain of independent systems of equations for each component $\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x},t)$. In the preceding example such a "decomposition" was in fact realized, but only thanks to the highly special form of the right side of the equation under consideration. In the general case the possibility of such a decomposition occurs only under the following rather rigid requirements.

Let the function $h(\mathbf{x},t)$ be given as a product

$$h(\mathbf{x},t) = \mu(t)H(\mathbf{x},t),$$

where $\mu(t)$ is some smooth function of time t and $H(\mathbf{x},t)$ is the first integral (possibly trivial) of the (full) system considered. In this case the vector function $\mathbf{f}_{q+\chi m}(\mathbf{x},t)$ satisfying the chain of equations

$$\begin{aligned} &\left[\mathbf{f}_{\mathrm{q}+\chi m}, \mathbf{g}\right](\mathbf{x},t) - \mu(t)H(\mathbf{x},t)\tfrac{\partial \mathbf{f}_{\mathrm{q}+\chi m}}{\partial t}(\mathbf{x},t) + \\ &+ (\mu(t)H(\mathbf{x},t) + m\beta)\,\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x},t) = 0, \quad m = 1,2,\ldots, \end{aligned} \tag{1.139}$$

likewise satisfies system (1.133).

However, this approach assumes an a priori knowledge of one of the integrals of the full system, which as a rule isn't available. So, in the rare exceptional case, under the action of an generalized symmetry group, time is transformed independent of the phase variables, i.e. it is happens that $H(\mathbf{x}, t) \equiv 0$. If in this the required vector functions $\mathbf{f}_{q+\chi m}$ are considered independent of time, then μ automatically becomes a linear function of t, $\mu = \nu t + \mu_0$, and (1.139) is rewritten in the form

$$\left[\mathbf{f}_{q+\chi m}, \mathbf{g}\right](\mathbf{x}) = (\mu\beta + \nu)\mathbf{f}_{q+\chi m}(\mathbf{x}) = 0, \quad m = 1, 2, \ldots \tag{1.140}$$

This shows that $\mathbf{g}$ generates a generalized symmetry group for each vector of the field $\mathbf{f}_{q+\chi m}$.

We note the connection of the theory of exponentially-asymptotic generalized symmetry groups with semi-quasihomogeneous systems:

Example 1.5.4. The method of constructing particular solutions of semi-quasihomogeneous systems investigated in the preceding section can be interpreted from the group-theoretical viewpoint. So again we consider some semi-quasihomogeneous system (1.3). If in the Fuchsian system of equations (1.10) we make the obvious change of variable $\sigma = -\ln \mu$, this system is transformed to the constant coefficient system of linear equations

$$\frac{d\mathbf{x}}{d\sigma} = -\mathbf{G}\mathbf{x}, \quad \frac{dt}{d\sigma} = t. \tag{1.141}$$

It is of course understood that (1.141) generates the generalized symmetry group of the truncated system (1.9) and the exponentially-asymptotic generalized symmetry group of the full system (1.3). Particular solutions of the truncated system, lying on an orbit of this group, should thus be sought in the form

$$\mathbf{x} = \exp(-\mathbf{G}\sigma)\mathbf{y}_0, \quad t = e^{\sigma}\tau_0,$$

which is equivalent to

$$\mathbf{x} = (t/\tau_0)^{-\mathbf{G}}\, y_0.$$

From (1.130) it follows that the quantities $\mathbf{y}_0$ and τ_0 must satisfy the equality

$$\tau_0 \mathbf{f}_q(\mathbf{y}_0, \tau_0) = -\mathbf{G}\mathbf{y}_0, \tag{1.142}$$

analogous to (1.13).

A substitution of the type (1.123), generating the system of linear equations (1.141), converts system (1.3) to the system

$$\frac{d\mathbf{y}}{d\sigma} = \mathbf{G}\mathbf{y} + \tau \sum_{m=0} e^{-m\beta\sigma}\mathbf{f}_{q+\chi m}(\mathbf{y}, \tau). \tag{1.143}$$

From this it is partly evident that, by satisfying (1.142) for some fixed $(\mathbf{y}_0, \tau_0) \in \Omega$, the matrix $\mathbf{K}$ has the form

$$\mathbf{K} = \mathbf{G} + \tau_0 d_{\mathbf{y}} \mathbf{f}_{\mathrm{q}}(y_0, \tau_0)$$

and coincides with the Kovalevsky matrix if we set $\tau_0 = \gamma$, $\mathbf{y}_0 = \mathbf{x}_0^{\gamma}$.

System (1.143) has a particular solution $\mathbf{y}(\sigma)$, represented in the form of a series

$$\mathbf{y}(\sigma) = \sum_{k=0}^{\infty} \mathbf{y}_k(\sigma) e^{-k\beta\sigma},$$

analogous to (1.19).

This means that the original system (1.3) has a solution in the form of a series (1.16).

With the help of the theory of exponentially-asymptotic symmetry groups it is likewise possible to construct solutions converging to singular points with unbounded increase or decrease of the independent variable and represented in the form of hybrid Lyapunov series [133], containing exponentials, and of series of type (1.16).

Example 1.5.5. We consider the two-dimensional system of equations

$$\dot{x} = -\left(1 + \sum_{m=2}^{\infty} a_m y^m\right) x, \quad \dot{y} = -y^2. \tag{1.144}$$

It has an obvious family of solutions converging to $x = y = 0$ as $t \to +\infty$:

$$\begin{aligned} x(t) &= ce^{-t} \prod_{m=2}^{\infty} \exp\left(\tfrac{a_m}{m-1} t^{1-m}\right) = ce^{-t}\left(1 + \textstyle\sum_{k=1}^{\infty} x_k t^{-k}\right), \\ y(t) &= t^{-1}, \end{aligned} \tag{1.145}$$

where the coefficients x_k depend polynomially on the a_m.

We consider the group of transformations of three-dimensional phase space, generating the following system of equations:

$$\frac{dx}{d\sigma} = -tx, \ \frac{dy}{d\sigma} = -y, \ \frac{dt}{d\sigma} = t. \tag{1.146}$$

System (1.146) is easily interpreted. The flow it generates has the form:

$$x = \exp\left(\tau(1 - e^{\sigma})\right)\xi, \quad y = e^{-\sigma}\eta, \quad t = e^{\sigma}\tau. \tag{1.147}$$

We consider the change of variables (1.147), which converts (1.144) into the form

$$\xi' = -\left(1 + \sum_{m=2} a - me^{(1-m)\sigma}\eta^m\right)\xi, \quad \eta' = -\eta^2,$$

where $(\cdot)' = \frac{d}{d\tau}$.

Consequently, the flow (1.147) of the system of differential equations (1.146) is exponentially asymptotic to the symmetry group of system (1.144). The truncated system

$$\dot{x} = -x, \ \dot{y} = -y^2$$

has the one-parameter family of particular solutions

$$x = ce^{\tau_0 - t}, \ y = y_0(\frac{t}{\tau_0})^{-1}, \ y_0 = \tau_0^{-1},$$

where c is the parameter of the family and τ_0 is fixed, lying on the orbit of the group (1.147).

Thus, in agreement with Theorem 1.5.2, the family of particular solutions (1.145) for system (1.144) generates the group (1.147).

More details about the analysis of hybrid series containing negative powers of e^t and t are left for Chap. 3.

Chapter 2
The Critical Case of Pure Imaginary Roots

2.1 Asymptotic Solutions of Autonomous Systems of Differential Equations in the Critical Case of m Pairs of Pure Imaginary and $n - 2m$ Zero Roots of the Characteristic Equation

In the preceding chapter we investigated asymptotic solutions of smooth systems of differential equations as $t \to \pm 0$, or as $t \to \pm\infty$, where it wasn't stipulated that the desired solutions be located in a neighborhood of a critical point. In this chapter we will examine closely the existence of solutions that converge to a critical point as $t \to +\infty$ or as $t \to -\infty$.

The principal object of investigation in this section will be an infinitely smooth system of differential equations for which the origin $\mathbf{x} = \mathbf{0}$ is a critical point and the right side is developed in a Maclaurin series:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f}(0) = 0, \quad \mathbf{x} \in \mathbb{R}^n. \tag{2.1}$$

In the preceding chapter, for immediate application of the techniques developed to the problem of finding asymptotic solutions, it was actually required that all roots of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{E}) = 0$ be zero, where $\mathbf{A} = d\mathbf{f}(0)$ is the Jacobian matrix of the vector field $\mathbf{f}(\mathbf{x})$ evaluated at the critical point $\mathbf{x} = 0$. We will replace this requirement by something more natural. If we consider the system on the invariant center manifold, then the characteristic equation for the reduced system will have only purely imaginary roots, among which some zeros can of course appear. We will therefore stipulate that the characteristic equation of system (2.1) have m pairs of purely imaginary roots, the remaining $n - 2m$ roots being zero.

Further, we will need some facts from the theory of Poincaré normal forms [7, 41]. We consider some formal coordinate transformations $\mathbf{x} \mapsto \mathbf{y}$ using Maclaurin series in the neighborhood of $\mathbf{y} = 0$ without changing the linear part:

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$$\mathbf{x} = \mathbf{y} + \sum_{m=2}^{\infty} \mathbf{Y}_m(\mathbf{y}), \tag{2.2}$$

where $\mathbf{Y}_m$ is a homogeneous vector function of degree m.

After the substitution (2.2), system (2.1) assumes the form

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}), \quad (0) = 0, \quad d\mathbf{g}(0) = \mathbf{A}. \tag{2.3}$$

The substitution (2.2) is chosen so that the system (2.3) will have the simplest form.

We represent the matrix $\mathbf{A}$ as the sum of two matrices $\mathbf{D} + \mathbf{J}$, where the matrix $\mathbf{D}$ is diagonalizable and the matrix $\mathbf{J}$ is equivalent to a block-Jordan matrix with zero diagonal. We express the right side of Eq. (2.3) thus:

$$\mathbf{g}(\mathbf{y}) = \mathbf{D}\mathbf{y} + \mathbf{h}(\mathbf{y}), \quad \mathbf{h}(\mathbf{y}) = \mathbf{J}\mathbf{y} + \dots,$$

where the dots represent the totality of nonlinear terms.

Definition 2.1.1. We say that Eq. (2.3) has *Poincaré normal form* if, for arbitrary $\mathbf{z} \in \mathbb{R}^n, \sigma \in \mathbb{R}$, the formal equality

$$\exp(-\mathbf{D}\sigma)\mathbf{h}\,(\exp(\mathbf{D}\sigma)\mathbf{z}) = \mathbf{h}(\mathbf{z}) \tag{2.4}$$

holds.

Formula (2.4) is extremely useful in applications, but using it to check the normality of (2.3) is quite a difficult matter. Therefore in practice we ordinarily employ another method. The relation (2.4) indicates that (2.3) has a symmetry group, generated by the linear system of equations

$$\frac{d\mathbf{y}}{d\sigma} = \mathbf{D}\mathbf{y}. \tag{2.5}$$

If all the eigenvalues of the matrix $\mathbf{A}$—and consequently also of $\mathbf{D}$—are pure imaginary, then (2.5) generates some group of rotations of phase space.

From this we immediately get a rule for checking normality: the vector field $\mathbf{h}(\mathbf{y})$ must satisfy the system of equations

$$[\mathbf{D}\mathbf{y}, \mathbf{h}(\mathbf{y})] \equiv 0. \tag{2.6}$$

But Eq. (2.6) also doesn't give a clear idea of the normal form of the right side of the system.

We suppose that the matrix $\mathbf{D}$ has been reduced to diagonal form

$$\mathbf{D} = \text{diag}\,(\lambda_1, \dots, \lambda_n).$$

Here, in general, we need to take into account that, as a result of such a reduction, the right sides of (2.1) and (2.3) can be complex. We consider the expansion of the j–th component of the vector field $\mathbf{h}(\mathbf{y})$ in a Maclaurin series:

$$h^j = \sum_{p_1,\dots,p_n} h^j_{p_1,\dots,p_n}(y^1)^{p_1}\dots(y^n)^{p_n}.$$

We consider the vectors

$$\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n,\ \mathbf{p} = (p_1,\dots,p_n) \in \mathbb{N}_0^n.$$

By the norm of a vector we understand the quantity

$$|\mathbf{b}| = |b_1| + \dots + |b_n|.$$

Definition 2.1.2. We say that the monomial

$$h^j_{p_1,\dots,p_n}(y^1)^{p_1}\dots(y^n)^{p_n},\ |\mathbf{p}| \geq 2,$$

is *resonant* if

$$\lambda_j = \langle\boldsymbol{\lambda},\mathbf{p}\rangle. \tag{2.7}$$

Relation (2.7) is called a resonance of order $p = |\mathbf{p}|$

Definition 2.1.3. We say the system (2.3) is written in Poincaré normal form if the expansion of the nonlinear part of the vector field $\mathbf{h}(\mathbf{y})$ into a Maclaurin series contains only resonant monomials.

It can be shown that definitions (2.1) and (2.3) are equivalent [7, 41]. The following important result holds [7,41].

Theorem 2.1.1 (Poincaré-Dulac). *By use of the formal expansion (2.2), system (2.1) can always be brought into the Poincaré normal form (2.3).*

The reduction of a system to normal form greatly simplifies investigation of the behavior of its trajectories in the neighborhood of a critical point, at least on a formal level. The motion of a system written in normal form admits formal dichotomy, i.e. if $\mathbf{J} = \mathbf{0}$ then the solution of system (2.3) is obtained with the aid of the superposition of solutions of the linearized system and the system containing only nonlinear terms. A more precise expression of this result is given by the following lemma on the dichotomy of motions.

Lemma 2.1.1 ([140]). *The linear change of variables*

$$\mathbf{y} = \exp(\mathbf{D}t)\mathbf{z} \tag{2.8}$$

transforms the normalized system of equations (2.3) into the system

$$\dot{\mathbf{z}} = \mathbf{h}(\mathbf{z}). \tag{2.9}$$

Proof. The lemma is a direct consequence of Lemma 1.5.4

Remark 2.1.1. The matrix $\mathbf{J}$, being the Jacobi matrix of the right side of system (2.9) evaluated at the point $\mathbf{z} = 0$, is like a block-Jordan matrix with zero diagonal. If not all entries of this matrix are zero, then (2.9) allows further simplification connected with reduction to the so-called Belitskiy normal form [13]. However, when later on it comes to applying conditions obtained for the existence of asymptotic solutions in concrete examples, it will suffice for us to use the Poincaré normal form, so we won't need to explain the Belitskiy normal form.

Of course, the development of the theory of normal forms isn't just applied in the case of all purely imaginary roots of the characteristic equation. However, this theory is applied most effectively in critical cases, e.g. in the study of systems reduced on the center manifold. System (2.9) has a form that is suitable for investigations that use the methods explained in the preceding chapter, so long as the characteristic equation of its linear part has only zero roots. Therefore, from the formal point of view, the program of searching for an asymptotic solution of the original system (2.1) consists of finding a suitable group of quasihomogeneous dilations, on the whose orbits would be located a particular asymptotic solution of the system, truncated with the help of the introduced structure. If such a procedure should achieve success, then the solution of the full system (2.9) can be written in the customary form, analogous to (1.16):

$$\mathbf{z}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{z}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta}.$$

Since the eigenvalues of the matrix $\mathbf{D}$ lie on the imaginary axis, transformation (2.8) leads only to a "twisting" of the constructed solution, not altering its convergence to the critical point. The asymptotic solution of the normalized system (2.3) could then be written in the form

$$\mathbf{y}(t) = \exp\left(\mathbf{D}t - \mathbf{G}\ln(\gamma t)\right) \sum_{k=0}^{\infty} \mathbf{z}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta}.$$

However, the actual situation turns out to be much more complicated. In a typical case, the Maclaurin series (2.2) converges in some neighborhood of $\mathbf{y} = \mathbf{0}$, at least in the absence of resonances (2.7) and with satisfaction of some additional Diophantine conditions on the eigenvalues of the matrix $\mathbf{A}$ [7, 30, 32]. There are rather many results showing that, by violating these conditions, the normalizing transformation may diverge [85, 86]. For this reason the right sides of systems (2.3) and (2.9) are only formal series, most likely divergent. Although, according to Borel's theorem [142], there exist smooth systems of differential equations for which these series will be the Maclaurin series of the right sides, a transfer of properties of solutions of these systems to the original system (2.1) requires some additional justification.

Nonetheless, the following assertion is true and generalizes a theorem from [62]:

Theorem 2.1.2. *Suppose that all eigenvalues of the matrix* $\mathbf{A}$ *are pure imaginary or zero, and let system (2.9) be positive semi-quasihomogeneous with respect to the structure given by the real matrix* $\mathbf{G}$*, whose eigenvalues have positive real parts. Let the quasihomogeneous truncation*

$$\dot{\mathbf{z}} = \mathbf{h}_{\mathrm{q}}(\mathbf{z}) \tag{2.10}$$

of system (2.9) be such that there is a vector $\mathbf{z}_0^{\gamma} \in \mathbb{R}^n$, $\mathbf{z}_0^{\gamma} \neq 0$, *and a number* $\gamma = \pm 1$ *satisfying the algebraic system of equations*

$$-\gamma \mathbf{G} \mathbf{z}_0^{\gamma} = \mathbf{h}_{\mathrm{q}}(\mathbf{z}_0^{\gamma}). \tag{2.11}$$

Then the system of differential equations (2.1) has an asymptotic solution $\mathbf{x}(t) \to 0$ *as* $t \to \gamma \times \infty$.

Proof. We first reduce system (2.1) to the formal normal form (2.3) and we consider the truncated system (2.10). In the theory of critical cases, such truncated normal forms are also known as model systems. It is clear that in order to choose a model system it suffices to know just a finite number of terms of the Maclaurin expansion of $\mathbf{h}(y)$ and, consequently, just a finite number of forms $\mathbf{Y}_2, \ldots, \mathbf{Y}_M$ in the normalizing series (2.2).

Now, instead of a complete normalization, we proceed piecewise by the formula

$$\mathbf{x} = \mathbf{y} + \sum_{m=2}^{M_*} \mathbf{Y}_m(\mathbf{y}),$$

where $M_* > M$ must thus be taken sufficiently large.

Of course, this transformation will be analytic in some neighborhood of the critical point $\mathbf{y} = 0$. After the substitution (2.8), the system of equations under investigation assumes the form

$$\dot{\mathbf{z}} = \mathbf{h}(\mathbf{z}, t), \tag{2.12}$$

where the right side is represented in the form

$$\mathbf{h}(\mathbf{z}, t) = \mathbf{h}_{\mathrm{q}}(\mathbf{z}) + \mathbf{h}^*(\mathbf{z}) + \mathbf{h}^{**}(\mathbf{z}, t),$$

discussed in Remark 1.3.4. Since the roots of the linear approximation system are purely imaginary, the $\mathbf{h}^{**}(\mathbf{z}, t)$ term, not “fitting into” the quasihomogeneous scale, depends on time periodically or quasiperiodically. Thus Theorems 1.3.2 and 1.3.3 are applied to the system (2.12). Since the truncated system is autonomous and the equality (2.11) holds, the linear system (1.41) will likewise be autonomous and will consequently also hold. Thus the system of equations (2.12) has the particular solution

$$\mathbf{z}(t) = (\gamma t)^{-\mathbf{G}} \left(\mathbf{z}_0^{\gamma} + o(1)\right) \quad \text{as} \quad t \to \gamma \times \infty.$$

Since the real parts of all eigenvalues of the matrix $\mathbf{G}$ are strictly positive and $\chi = 1$, we have $\mathbf{z}(t) \to 0$ as $t \to \gamma \times \infty$.

Returning to the original variable $\mathbf{x}$, we obtain the required assertion.

The theorem is proved.

The theorem just proved has important applications to the theory of the stability of motion.

Theorem 2.1.3. *Let the system of first approximation for (2.1) have only pure imaginary or zero roots, and let its normal form be such that the system obtained from it by discarding the diagonalized linear terms is positive semi-quasihomogeneous with respect to the structure given by some matrix* $\mathbf{G}$*, whose eigenvalues have only positive real part. If, for the vector field* $\mathbf{h}_\mathrm{q}$*, there exists a vector* $\mathbf{z}_0^- \in \mathbb{R}^n$*,* $\mathbf{z}_0^- \neq 0$*, such that*

$$\mathbf{G}\mathbf{z}_0^- = \mathbf{h}_\mathrm{q}(\mathbf{z}_0^-)$$

(i.e. $\gamma = -1$*), then the critical point of* $\mathbf{x} = \mathbf{0}$ *of system (2.1) is unstable.*

The proof follows from the existence of a particular solution $\mathbf{x}$ of system (2.1) such that $\mathbf{x}(t) \to 0$ as $t \to -\infty$.

Thus we again encounter the problem of finding nonzero eigenvectors of a quasihomogeneous vector field. Their existence is largely dependent on the manner of choosing the quasihomogeneous structure. First, one and the same system of equations can be semi-quasihomogeneous with respect to various structures and, as may already be evident from concrete examples, frequently not all the corresponding truncated systems have the necessary particular solutions. Second, one and the same model system can yield different structures, for which the corresponding transformation groups have distinct orbits (see Example 1.1.2). In all this, the particular solution with the asymptotic that interests us may lie on the orbit of but one transformation from the group. It should be noted that a more natural way of finding quasihomogeneous structures is through analysis of Newton manifolds. How should we proceed if the proposed truncation, gotten with the help of these techniques, turns out not to have eigenvectors? One of the possible recipes consists in attempting to construct a particular solution of the given model system in the form (1.36), followed by application of Theorem 1.3.2. But this problem reduces to finding a bounded particular solution $\mathbf{x}_0^{\gamma}(\cdot)$ for the system of differential equations (1.40), which might not be integrable by quadratures. In order to stay within the framework of solutions of algebraic—rather than of differential—equations, we need to attempt to find a different structure, giving the very same truncation, and consider the problem of constructing eigenvectors with respect to this new structure. The apparatus for reduction to normal form indicates a concrete method of searching for such algebraic quasihomogeneous structures; this method is based on the following assertion:

Lemma 2.1.2. *Consider a system of differential equations of form (2.9) that is semi-quasihomogeneous with respect to the structure generated by some matrix* **G***, and consider the expansion of its right side in a series of quasihomogeneous forms:*

$$\mathbf{h}(\mathbf{z}) = \sum_{m=0} \mathbf{h}_{q+\chi m}(\mathbf{z}).$$

Let a linear system of differential equations of type (2.5) generate the symmetry group of system (2.9), where the matrices **G** *and* **D** *commute:* $\mathbf{GD} = \mathbf{DG}$. *Then (2.9) is semi-quasihomogeneous with respect to the set of structures generated by the one-parameter family of matrices*

$$\mathbf{G}_\delta = \mathbf{G} + \delta\mathbf{D}, \tag{2.13}$$

and its right side has the very same expansion.

Proof. Since (2.9) is invariant with respect to the action of the flow of system (2.5), the following identity holds:

$$\exp(-\mathbf{D}\sigma)\mathbf{h}\left(\exp(\mathbf{D}\sigma)\mathbf{z}\right) = \mathbf{h}(\mathbf{z}).$$

We set $\mu = e^{\sigma/\delta}$. Using (1.17) and the fact that the exponential of the sum of commuting matrices equals the product of their exponentials, we get

$$\mathbf{h}\left(\mu^{\mathbf{G}_\delta}\mathbf{z}\right) = \mathbf{h}\left(\exp(\mathbf{D}\sigma)\mu^{\mathbf{G}}\mathbf{z}\right) = \exp(\mathbf{D}\sigma)\sum_{m=0}\mathbf{h}_{q+\chi m}(\mu^{\mathbf{G}}\mathbf{z}) =$$
$$= \mu^{\delta\mathbf{D}}\sum_{m=0}\mu^{\mathbf{G}+(m\beta+1)\mathbf{E}}\mathbf{h}_{q+\chi m}(\mathbf{z}) = \mu^{\mathbf{G}_\delta+\mathbf{E}}\sum_{m=0}\mu^{m\beta}\mathbf{h}_{q+\chi m}(\mathbf{z}).$$

The lemma is proved.

If system (2.9) has $n_0 < n$ independent symmetry groups generating systems of linear differential equations of form (2.5) with matrices $\mathbf{D}_1, \ldots, \mathbf{D}_{n_0}$, then the matrix that generates the quasihomogeneous structure can be written in the following form:

$$\mathbf{G}_\delta = \mathbf{G} + \sum_{j=1}^{n_0}\delta_j\mathbf{D}_j. \tag{2.14}$$

Here $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{n_0})$ is a set of arbitrary parameters.

So, if the diagonal part of a linear system commutes with the remaining terms of the normal form, then a suitable quasihomogeneous scale for system (2.9) can be selected with the help of the family of matrices (2.13), where **D** is the diagonalized portion of the matrix for the system of first approximation.

This circumstance substantially extends the possibilities for searching for particular asymptotic solutions of the model system. We suppose that there does not exist a nonzero vector $\mathbf{z}_0^\gamma$, satisfying (2.11) for $\gamma = \pm 1$, but that for any $\delta \neq 0$ it is possible to select a nonzero vector $\mathbf{z}_0^\gamma$ such that

$$-\gamma(\mathbf{G} + \delta\mathbf{D})\mathbf{z}_0^\gamma = \mathbf{h}_\mathrm{q}(\mathbf{z}_0^\gamma). \tag{2.15}$$

This means that, instead of a solution of simple form

$$\mathbf{z}^\gamma(t) = (\gamma t)^{-\mathbf{G}}\mathbf{z}_0^\gamma,$$

we will seek particular solutions of the model system (2.10) in the more complicated form

$$\mathbf{z}^\gamma(t) = \exp\left(\delta\mathbf{D}\ln t\right)(\gamma t)^{-\mathbf{G}}\mathbf{z}_0^\gamma.$$

Since all the eigenvalues of the matrix $\mathbf{D}$ are pure imaginary, its appearance in the role of a factor of the matrix $\exp(\delta\mathbf{D}\ln t)$ leads to a “twisting” of the quasihomogeneous ray.

By way of examples of the application of Theorems 2.1.2 and 2.1.3, we consider the problem of the stability of a critical point and the existence of asymptotic solutions of general four-dimensional systems of differential equations in the case where the roots of the characteristic equation of the first approximation system are pure imaginary. This problem is rather typical. Its *codimension*, without additional expressions, equals two: the critical case considered is realized in systems of equations whose right sides may depend, in the case of general position, on only two parameters (for details see the monograph [100]). We show that the existence of “twisted” rays under instability in the problem considered is the case of general position.

So, let the roots of the characteristic equation have the form:

$$\lambda_{1,2} = \pm i\omega_1, \quad \lambda_{3,4} = \pm i\omega_2,$$

where ω_1, ω_2 are frequencies of small vibrations.

Without loss of generality, we assume that the diagonalizing part of the first approximation matrix has the form:

$$\mathbf{D} = \begin{pmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{pmatrix}.$$

Example 2.1.1. Two pairs of purely imaginary roots in the absence of resonance between frequencies (Kamenkov’s problem [93]). Suppose the frequencies ω_1, ω_2 are not connected by any resonances up to the third order.

$$\omega_1 \neq \omega_2, \quad \omega_1 \neq 2\omega_2, \quad \omega_1 \neq 3\omega_2.$$

From this it follows that resonances of second order between the *roots of the characteristic equation of the first approximation system* are absent from the system and that the quadratic terms may be "killed" by means of a normalizing transformation.

In the phase space of the system we introduce Cartesian coordinates $\mathbf{z} = (x, u, y, v)$. Following [100], we write a cubic model system for the given problem:

$$\begin{aligned}
\dot{x} &= (x^2+u^2)(a_{11}x - b_{11}u) + (y^2+v^2)(a_{12}x - b_{12}u),\\
\dot{u} &= (x^2+u^2)(a_{11}u + b_{11}x) + (y^2+v^2)(a_{12}u - b_{12}x),\\
\dot{y} &= (x^2+u^2)(a_{21}y - b_{21}v) + (y^2+v^2)(a_{22}y - b_{22}v),\\
\dot{v} &= (x^2+u^2)(a_{21}v + b_{21}y) + (y^2+v^2)(a_{22}v + b_{22}u).
\end{aligned} \tag{2.16}$$

In the given example we will dwell but very briefly on the principles for constructing a model system. It is possible to show that the vector fields (2.16) (and only these) belong to the kernel of a linear operator commuting with the linear field $\mathbf{Dz}$ in the finite-dimensional space of cubic vector fields. The problem of explicitly describing the kernel of the given operator must be solved separately in every concrete case. For the solution of this problem, real Cartesian coordinates are not as a rule very convenient, and instead it is customary to use either complex conjugate Cartesian coordinates, for which the matrix $\mathbf{D}$ has diagonal form, or polar coordinates. In order to avoid having to write out tedious formulas for passing from one coordinate system to another, in the sequel we will only cite the final result and omit computational details.

It is clear that the model system has two independent symmetry groups, generated by the linear vector fields with the matrices

$$\mathbf{D}_1 = \begin{pmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{pmatrix}.$$

We will seek instability conditions for an original system of type (2.1) and, consequently, existence conditions for asymptotic solutions approaching its critical point $x = u = y = v = 0$ as $t \to -\infty$ in terms of the coefficients of the model system. In [100] conditions for asymptotic instability are described: the following conditions must be satisfied *simultaneously*

$$(1)\ a_{11} < 0, \qquad (2)\ a_{22} < 0,$$
$$(3)\ \text{ if } a_{12} > 0 \text{ and } a_{21} > 0, \text{ then } \Delta = a_{11}a_{22} - a_{12}a_{21} > 0.$$

Here all solutions of the original system, beginning in some small neighborhood of the critical point, are asymptotic as $t \to +\infty$. This result is obtained with the aid of construction of Lyapunov functions.

We show that by (strict) violation of sign in at least one of the conditions (1)–(3), the model system (2.16) will have an increasing solution of the type of a twisted linear ray. In this we can show that, in the general situation, (2.16) does not have particular solutions in the form of *linear* rays. Consequently the right side of (2.16), being a *homogeneous* vector field, must have a nonzero eigenvector with coordinates $(x_0^- u_0^-, y_0^-, v_0^-)$ for $\gamma = -1$ with respect to a matrix of type (2.14):

$$\mathbf{G}_\delta = \frac{1}{2}\mathbf{E} + \delta_1 \mathbf{D}_1 + \delta_2 \mathbf{D}_2. \tag{2.17}$$

Since the identity matrix commutes with an arbitrary matrix, system (2.16) is quasihomogeneous with respect to the structures generated by the matrices $\mathbf{G}_\delta$.

The existence of such a solution for the model system implies the instability of the original system.

(1) Let $a_{11} > 0$. Then (2.16) has a *one-parameter* family of particular solutions of the form:

$$\begin{aligned} x^-(t) &= \frac{1}{\sqrt{-t}}\left(x_0^- \cos(\delta\omega_1 \ln(-t)) - u_0^- \sin(\delta\omega_1 \ln(-t))\right), \\ u^-(t) &= \frac{1}{\sqrt{-t}}\left(x_0^- \sin(\delta\omega_1 \ln(-t)) + u_0^- \cos(\delta\omega_1 \ln(-t))\right), \\ y^-(t) &= 0, \quad v^-(t) = 0, \end{aligned}$$

where

$$x_0^-(t) = \rho\cos\theta, \quad u_0^-(t) = \rho\sin\theta, \quad \rho = \frac{1}{\sqrt{2a_{11}}},$$

and θ is a parameter of the family of solutions of the system of algebraic equations

$$\begin{aligned} &-\begin{pmatrix} 1/2 & -\delta\omega_1 \\ \delta\omega_1 & 1/2 \end{pmatrix}\begin{pmatrix} x_0^- \\ u_0^- \end{pmatrix} + \\ &+\begin{pmatrix} (x_0^-)^2 + (u_0^-)^2\, a_{11}x_0^- - b_{11}u_0^- \\ (x_0^-)^2 + (u_0^-)^2\, a_{11}u_0^- + b_{11}x_0^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{2.18}$$

for $\delta = \dfrac{b_{11}}{2a_{11}\omega_1}$.

The system (2.18) is analogous to system (2.15).

Here we set $\delta_1 = \delta_2 = \delta$ in formula (2.17), i.e. we seek the matrix of the required quasihomogeneous truncation in the form (2.13). Since the diagonalizing matrix $\mathbf{D}$ is the matrix of the first approximation system obtained from the formal normal form, the system of type (2.9) for the case considered is semi-quasihomogeneous with respect to the structure generated by the matrix $\mathbf{G}_\delta$. The existence of solutions of the truncated system of the indicated type implies the existence of particular solutions of the original system of type (2.1), asymptotic to the critical point as $t \to -\infty$, indicating instability.

(2) The case $a_{22} > 0$ is considered analogously. System (2.16) has the family of particular solutions:

$$
\begin{aligned}
&x^-(t) = 0, \quad u^-(t) = 0,\\
&y^-(t) = \frac{1}{\sqrt{-t}}\left(y_0^- \cos(\delta\omega_2 \ln(-t)) - v_0^- \sin(\delta\omega_2 \ln(-t))\right),\\
&v^-(t) = \frac{1}{\sqrt{-t}}\left(y_0^- \sin(\delta\omega_2 \ln(-t)) + v_0^- \cos(\delta\omega_2 \ln(-t))\right),
\end{aligned}
$$

where

$$
y_0^-(t) = \rho\cos\theta, \quad v_0^-(t) = \rho\sin\theta \quad \rho = \frac{1}{\sqrt{2a_{22}}}
$$

and θ is a parameter of the family that satisfies the system of algebraic equations

$$
\begin{aligned}
&-\begin{pmatrix} 1/2 & -\delta\omega_2 \\ \delta\omega_2 & 1/2 \end{pmatrix}\begin{pmatrix} x_0^- \\ u_0^- \end{pmatrix} + \\
&+\begin{pmatrix} (y_0^-)^2 + (v_0^-)^2 \; a_{22}y_0^- - b_{22}v_0^- \\ (y_0^-)^2 + (v_0^-)^2 \; a_{22}v_0^- + b_{22}y_0^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}
\tag{2.19}
$$

Since system (2.19) is likewise analogous to (2.15), we set $\delta = \dfrac{b_{22}}{2a_{22}\omega_2}$.

For demonstrating the existence of asymptotic solutions in cases (1) and (2), we used the quasihomogeneous structure in conformity with the recipe of Lemma 2.1.2. We could have proceeded otherwise: the asymptotic solution we found for the model system has, in fact, the form (1.36), where $\mathbf{G} = \frac{1}{2}\mathbf{E}$, so that in this case we might have applied Theorem 1.3.2. System (1.43) will be correct in the case considered, since its periodic coefficients depend on the logarithmic time τ.

(3) Now let $a_{11} < 0$, $a_{22} < 0$, $a_{12} > 0$, $a_{21} > 0$, with $\Delta = a_{11}a_{22} - a_{12}a_{21} < 0$. This case differs essentially from the preceding ones. System (2.16) possesses a two-parameter family of particular solutions of the form:

$$
\begin{aligned}
x^-(t) &= \frac{1}{\sqrt{-t}}\left(x_0^- \cos(\delta_1\omega_1 \ln(-t)) - u_0^- \sin(\delta_1\omega_1 \ln(-t))\right),\\
u^-(t) &= \frac{1}{\sqrt{-t}}\left(x_0^- \sin(\delta_1\omega_1 \ln(-t)) + u_0^- \cos(\delta_1\omega_1 \ln(-t))\right),\\
y^-(t) &= \frac{1}{\sqrt{-t}}\left(y_0^- \cos(\delta_2\omega_2 \ln(-t)) - v_0^- \sin(\delta_2\omega_2 \ln(-t))\right),\\
v^-(t) &= \frac{1}{\sqrt{-t}}\left(y_0^- \sin(\delta_2\omega_2 \ln(-t)) + v_0^- \cos(\delta_2\omega_2 \ln(-t))\right),
\end{aligned}
$$

where

$$x_0^- = \rho\cos\theta, \quad u_0^- = \rho\sin\theta, \quad v_0^- = \nu\rho\sin\varphi, \quad y_0^- = \nu\rho\cos\varphi,$$
$$\rho = \sqrt{\frac{a_{22}-a_{12}}{2\Delta}}, \quad \nu = \sqrt{\frac{a_{11}-a_{21}}{a_{22}-a_{12}}},$$
$$\delta_1 = \frac{(b_{11}+\nu^2 b_{12})\rho^2}{\omega_1}, \quad \delta_2 = \frac{(b_{21}+\nu^2 b_{22})\rho^2}{\omega_2}.$$

Here θ, φ are parameters of the family. It is worth noting that the quantities ρ and ν are real (from the assumptions on the coefficients a_{ij}).

Because of its complexity, we won't write out the system of algebraic equations that the quantities $x_0^-, u_0^-, y_0^-, v_0^-$ must satisfy.

As was already noted, the model system (2.16) is quasihomogeneous with respect to the structure generated by the matrix (2.17). The particular solutions in the two-parameter family given above lie on orbits of the corresponding transformation group. But a "full" system of type (2.9), obtained from the normal form by discarding linear terms, does not turn out to be a semi-quasihomogeneous structure with respect to this family. The presence of resonances of higher orders between the frequencies

$$\omega_1 = p\omega_2, \quad p \geq 4$$

may lead to the situation that the linear vector fields with matrices $\mathbf{D_1}, \mathbf{D_2}$ will no longer generate symmetry groups of the *full* system. Therefore, for proof of existence of an asymptotic solution as $t \to -\infty$ for the original system, it follows that we must apply a result of the type of Theorem 1.3.2. In the case considered a system of type (2.12) will be semi-(quasi)homogeneous with respect to the structure generated by the matrix $\frac{1}{2}\mathbf{E}$. The solution indicated above will have the form (1.36), whereby the linear system (1.43) will likewise be correct.

Example 2.1.2. Two pairs of pure imaginary roots with resonance 1 : 2 between the frequencies. Let ω_1, ω_2 be connected by the resonance relation

$$\omega_1 = 2\omega_2.$$

This problem has codimension 3. Here not all the quadratic terms can be killed by means of normalizing substitutions. Writing $\mathbf{z} = (x, y, u, v)$ and using results from [100], we write a model system that will be quadratic in the given case:

$$\dot{x} = a_1(y^2 - v^2) - 2b_1 yv, \qquad \dot{u} = b_1(y^2 - v^2) + 2a_1 yv,$$
$$\dot{y} = a_2(xy + uv) - b_2(uy - xv), \quad \dot{v} = 2_2(xy + uv) + a_2(uy - xv). \tag{2.20}$$

This is a homogeneous system with quadratic right sides. In case the critical point $x = u = y = v = 0$ of the model system (2.1.20) is isolated, then by Lemma 1.1.1 this system will have a linear particular solution. But (2.1.20) has a two-dimensional manifold of critical points: $y = v = 0$. Therefore we must look for an asymptotic solution (2.1.20) in the form of a twisted ray.

In the monograph [100] it is shown that, in the general situation, (2.1.20) possesses two one-parameter families of particular solutions of the form:

$$\begin{aligned} x^-(t) &= (-t)^{-1}\left(x_0^- \cos(2\delta\omega \ln(-t)) - u_0^- \sin(2\delta\omega \ln(-t))\right), \\ u^-(t) &= (-t)^{-1}\left(x_0^- \sin(2\delta\omega \ln(-t)) + u_0^- \cos(2\delta\omega \ln(-t))\right), \\ y^-(t) &= (-t)^{-1}\left(y_0^- \cos(2\delta\omega \ln(-t)) - v_0^- \sin(2\delta\omega \ln(-t))\right), \\ v^-(t) &= (-t)^{-1}\left(y_0^- \sin(2\delta\omega \ln(-t)) + v_0^- \cos(2\delta\omega \ln(-t))\right), \end{aligned}$$

where

$$x_0^- = \rho\cos\theta,\ u_0^- = \rho\sin\theta,\ y_0^- = \nu\rho\cos\varphi,\ v_0^- = \nu\rho\sin\varphi,\ \omega = \omega_2.$$

In this, the resonance phase $\psi = \theta - 2\varphi$ remains fixed, so that we may take both θ and φ as parameters of the family. We define constants ρ, ψ, ν, δ:

$$\begin{aligned} \rho = \nu^{-1}\sqrt{\frac{3\delta\omega}{l_1 l_2 \sin\alpha}}, \quad \psi &= \pi k - \alpha_2 + \arctan(\delta\omega), \quad k = 0, 1, \\ \nu &= \sqrt{\frac{l_2}{l_1}\left(\sqrt{8+\cos^2\alpha} - \cos\alpha\right)}, \\ \delta &= \frac{1}{4\omega\sin\alpha}\left(\sqrt{8+\cos^2\alpha} - 3\cos\alpha\right), \end{aligned}$$

where we have introduced the notations:

$$a_i = l_i \cos\alpha_1, \quad b_i = l_i \sin\alpha_1, \quad i = 1, 2.$$

From the expressed relations it follows that such a particular solution exists if

$$l_1 \neq 0, \quad l_2 \neq 0, \quad \alpha \neq \pi.$$

A system of type (2.15), which must be satisfied by the quantities x_0^-, u_0^-, y_0^-, v_0^-, is rather clumsy. For this reason we won't write it out explicitly.

The cases $\alpha = 0,\ \pi$ must generally be considered separately. With $\alpha = 0$ the constructions of two families of *twisted* solutions merge into one *straight line*:

$$\rho = \frac{1}{l_2}, \quad \psi = \pi - \alpha_2, \quad \nu = \sqrt{\frac{l_2}{l_1}}, \quad \delta = 0.$$

With $\alpha = 0$, the system (2.1.20) doesn't have asymptotic solutions as $t \to -\infty$.

Thus, with fulfillment of the indicated conditions, the model system admits solutions that are asymptotic to the critical point as $t \to -\infty$. These lie on the orbits of the group of transformations of extended phase space given by a linear system of equations of type (1.10), where $\mathbf{G} = \mathbf{G}_\delta = \mathbf{E} + \delta\mathbf{D}$. System (2.1.20) is quasihomogeneous with respect to the given structure, and the corresponding *full* system is semi-quasihomogeneous. Therefore we can use Theorem 2.1.3 to prove instability for the singular point of the original system.

Example 2.1.3. Resonance of frequencies 1: 1 in case of nonsimple elementary divisors. As before, we denote by **z** the four-dimensional vector (x, u, y, v). In the monograph [100], a model system is written for the given case. This system has the form

$$\begin{aligned}\dot{x} &= y, & \dot{u} &= v,\\ \dot{y} &= (ax - bu)(x^2 + u^2), & \dot{v} &= (ax + bu)(x^2 + u^2)\end{aligned} \tag{2.21}$$

System (2.21) is clearly quasihomogeneous, with matrix

$$\mathbf{G} = \operatorname{diag}(1, 1, 2, 2)$$

commuting with the diagonalized part of the matrix of the linear approximation

$$\mathbf{D} = \begin{pmatrix} 0 & -\omega & 0 & 0\\ \omega & 0 & 0 & 0\\ 0 & 0 & 0 & -\omega\\ 0 & 0 & \omega & 0 \end{pmatrix}.$$

Therefore the representations for this case of the analogs to systems (2.9) and (2.12) are semi-quasihomogeneous with respect to the structure generated by the family of matrices $\mathbf{G}_\delta = \mathbf{G} + \delta\mathbf{D}$. It is not difficult to verify that (2.21) doesn't have particular solutions in the form of quasihomogeneous rays. We will look for particular solutions of this system in the form of *twisted* rays. In this regard we note that, because of the invertibility of the equations of the model system, the existence conditions for such solutions will be the same for both $\gamma = +1$ and for $\gamma = -1$, so we will drop the lower index.

Thus

$$\begin{aligned} x(t) &= t^{-1}\left(x_0 \cos(\delta\omega \ln t) - u_0 \sin(\delta\omega \ln t)\right),\\ u(t) &= t^{-1}\left(x_0 \sin(\delta\omega \ln t) + u_0 \cos(\delta\omega \ln t)\right),\\ y(t) &= t^{-2}\left(y_0 \cos(\delta\omega \ln t) - v_0 \sin(\delta\omega \ln t)\right),\\ v(t) &= t^{-2}\left(y_0 \sin(\delta\omega \ln t) + v_0 \cos(\delta\omega \ln t)\right). \end{aligned}$$

The system of algebraic equations (2.15) is then written in the form

$$\begin{gathered} \begin{pmatrix} 1 & -\delta\omega & 0 & 0\\ \delta\omega & 1 & 0 & 0\\ 0 & 0 & 2 & -\delta\omega\\ 0 & 0 & \delta\omega & 2 \end{pmatrix} \begin{pmatrix} x_0\\ u_0\\ y_0\\ v_0 \end{pmatrix} + \\ + \begin{pmatrix} y_0\\ v_0\\ (ax_0 - bu_0)\left(x_0^2 + u_0^2\right)\\ (au_0 + bx_0)\left(x_0^2 + u_0^2\right) \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0 \end{pmatrix} \end{gathered} \tag{2.22}$$

System (2.22) has a *quasihomogeneous* family of solutions

$$x_0 = \rho\cos\theta, \qquad u_0 = \rho\sin\theta,$$
$$y_0 = \rho(\delta\omega\sin\theta - \cos\theta), \; v_0 = -\rho(\delta\omega\cos\theta + \sin\theta),$$

where θ is a parameter for the family and the quantities $\delta, \rho > 0$ satisfy the system of equations

$$a\rho^2 + (\delta\omega)^2 - 2 = 0, \quad 3\delta\omega - b\rho^2 = 0.$$

In complex form the conditions for the solvability of this last system take the following form:

$$c \neq -|c|, \quad \text{where} \quad c = a + ib,$$

i.e. they agree with the conditions for instability found in [100].

Therefore, under fulfillment of these conditions, the original full system of differential equations has particular solutions that converge to the critical point both as $t \to +\infty$ and as $t \to -\infty$. The existence of solutions asymptotic to the critical point as $t \to -\infty$ of course indicates unstable equilibrium for the original system.

We should therefore note that, in Examples 2.1.2 and 2.1.3, instability is the norm in that, in the parameter space of the model system, the set of values for which the conditions for the existence of asymptotic solutions is violated has measure zero.

The existence conditions for asymptotic solutions introduced in the preceding examples are described in the monograph [100] as conditions for instability. For this, instability is proved using the generalized Chetaev method: the authors prove that, in the neighborhood of an asymptotic solution for the model system, there exists some cone such that an arbitrary trajectory, beginning in this cone, leaves it through the base. With the help the Theorem 2.1.2 we showed that these very conditions guarantee the existence of asymptotic solutions for the original system as $t \to -\infty$, and we estimated the dimension of the set of these solutions.

For a more or less complete investigation of the problems posed, it would further be necessary to investigate the resonance $1:3$ (problem of codimension 3) and the resonance $1:1$ in the case of simple elementary divisors. It is interesting that the latter problem has a higher level of degeneracy: its codimension equals 4. These two problems, however, possess an unpleasant peculiarity: problems of stability for the given systems of differential equations are not algebraically solvable [98, 169]. This means roughly the following: domains of asymptotic stability or instability in the parameter space of a system have positive measure, but the hypersurface that is the boundary between these two domains is transcendental. Nonetheless, for resonance $1:3$ there exist sets of sufficient conditions for asymptotic stability and instability [97, 99], distinguishing subdomains with nonintersecting boundaries. Also interesting are results obtained relatively recently by P.S. Krasilnikov [120–122]: regardless of the algebraic insolubility in the parameter spaces for both problems, there exists a subset of positive measure where the criteria for asymptotic stability can be written explicitly, i.e. the interface of the asymptotically stable and unstable domains contains algebraic pieces. Nonetheless, the conditions obtained in those problems are exceedingly complicated and we will not consider them here. Things are still more complicated in problems of high codimension, where algebraic insolubility is the norm [5, 84].

2.2 Periodic and Quasiperiodic Systems

In the preceding section we obtained sufficient conditions for the existence of asymptotic solutions with nonexponential asymptotic for certain smooth autonomous systems of differential equations. The goal of this section to find analogous conditions for certain classes of nonautonomous systems of equations.

We consider a system of differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{f}(\mathbf{0}, t) \equiv \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{2.23}$$

with the assumption that $\mathbf{f}(\mathbf{x}, t)$ is a smooth vector function of $\mathbf{x}$, whose expansion into a Maclaurin series in the neighborhood of $\mathbf{x} = \mathbf{0}$ has coefficients that are smooth bounded functions of the time t. We begin with the simple case where $\mathbf{f}(\mathbf{x}, t)$ depends 2π–periodically on t:

$$\mathbf{f}(\mathbf{x},\ t + 2\pi) = \mathbf{f}(\mathbf{x},\ t).$$

We show that the question of existence of asymptotic solutions with nonexponential asymptotic for such periodic systems reduces, as in the autonomous case, to the question of existence of an eigenvector of a certain quasihomogeneous vector field.

We let $\mathbf{A}(t)$ denote the Jacobian matrix for the right side of system (2.23), evaluated at the origin ($\mathbf{A}(t) = d_x\mathbf{f}(0, t)$), and consider the linearized system of equations

$$\dot{x} = \mathbf{A}(t)\mathbf{x}. \tag{2.24}$$

First of all, in order to simplify the problem considered, we apply the well known Floquet-Lyapunov theory.

Theorem 2.2.1. *There exists a complex linear nondegenerate continuous 2π–periodic transformation of coordinates*

$$\mathbf{x} \mapsto \mathbf{B}(t)\mathbf{x}, \quad \mathbf{B}(t + 2\pi) \equiv \mathbf{B}(t), \quad \det \mathbf{B}(t) \neq 0,$$

that takes system (2.24) to autonomous form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{2.25}$$

Below we apply the above transformation to the nonlinear system (2.23). The system thus obtained will be, generally speaking, complex. However, the very same

Floquet-Lyapunov theory asserts that reduction of a linear system with 2π–periodic coefficients can be realized with the help of *real* 4π–periodic substitutions. The *nonlinear* system thereby obtained will likewise be 4π–periodic, but it can be made 2π–periodic by a change in the time scale. For reasons already given, we may assume without loss of generality that the linear approximation system for (2.23) is autonomous: $\mathbf{A}(t) \equiv \mathbf{A}$.

We assume that all roots $\lambda_1, \ldots, \lambda_n$ of the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{E}) = 0$ for system (2.25) are pure imaginary or zero.

The numbers $e^{2\pi\lambda_1}, \ldots, e^{2\pi\lambda_n}$ are called *multipliers* of system (2.24). If $\lambda_1, \ldots, \lambda_n$ lie on the imaginary axis, then the multipliers will lie on the unit circle.

We investigate the possibility of simplifying system (2.23) by means of a formal 2π–periodic t–dependent transformation:

$$\mathbf{x} = \mathbf{y} + \sum_{m=2}^{\infty} \mathbf{Y}_m(\mathbf{y}, t), \tag{2.26}$$

where the $\mathbf{Y}_m$ are homogeneous vector functions of degree m that are 2π–periodic: $\mathbf{Y}_m(\mathbf{y}, t+2\pi) \equiv \mathbf{Y}_m(\mathbf{y}, t)$.

After the transformation (2.26), system (2.23) is rewritten in the form of the system

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t), \quad \mathbf{y} \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad \mathbf{g}(\mathbf{0}, t) \equiv \mathbf{0}, \quad d_{\mathbf{y}} g(\mathbf{0}, t) = \mathbf{A}, \tag{2.27}$$

whose right side depends periodically on time: $\mathbf{g}(\mathbf{y}, t + 2\pi) \equiv \mathbf{g}(\mathbf{y}, t)$.

Again we represent the matrix $\mathbf{A}$ as the sum of two matrices $\mathbf{A} = \mathbf{D} + \mathbf{J}$, where the matrix $\mathbf{D}$ is diagonal and the matrix $\mathbf{J}$ is like a block-Jordan matrix, with null diagonal. We express the right side of (2.27) in the form

$$\mathbf{g}(\mathbf{y}, t) = \mathbf{D}\mathbf{y} + \mathbf{h}(\mathbf{y}, t), \quad \mathbf{h}(\mathbf{y}, t) = \mathbf{J}\mathbf{y} + \ldots,$$

where the dots represent terms that are nonlinear in $\mathbf{y}$.

As in the preceding section, we give two equivalent definitions of a normal form.

Definition 2.2.1. We say that system (2.27) is written in Poincaré normal form if, for any $\mathbf{z} \in \mathbb{R}^n$ and $\tau, \sigma \in \mathbb{R}$, the formal equation

$$\exp\left(-\mathbf{D}\sigma\right) \mathbf{h}\left(\exp\left(\mathbf{D}\sigma\right), \mathbf{z}, \tau + \sigma\right) = \mathbf{h}(\mathbf{z}, t). \tag{2.28}$$

holds.

System (2.27) clearly has a generalized symmetry group whose infinitesimal generator satisfies the system of equations

$$\frac{d\mathbf{y}}{d\sigma} = \mathbf{D}\mathbf{y}, \quad \frac{dt}{d\sigma} = 1. \tag{2.29}$$

Since all eigenvalues of the matrix **D** are pure imaginary, (2.29) generates a group of rotations of phase space and translations in time.

Using (1.128) we see that the vector function $\mathbf{h}(\mathbf{y}, t)$ must satisfy the system of partial differential equations

$$\frac{\partial \mathbf{h}}{\partial t}(\mathbf{y}, t) = (\mathbf{h}(\mathbf{y}, t),\ \mathbf{D}\mathbf{y})\,. \tag{2.30}$$

Nonetheless, neither the relation (2.28) nor the Eq. (2.30) concretely determines the form of the right side of the normalized system.

We write the matrix **D** in diagonal form:

$$\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

and consider the expansion of the j–th component of the vector function $\mathbf{h}(\mathbf{y}, t)$ in a Maclaurin series

$$h^j = \sum_{p_1,\ldots,p_n} h^j_{p_1,\ldots,p_n}(t)(y^1)^{p_1}\ldots(y^n)^{p_n},$$

where the coefficients $h^j_{p_1,\ldots,p_n}(t)$ are 2π–periodic in t.

We expand these functions in complex Fourier series and write a mixed expansion for h^j:

$$h^j = \sum_{p_1,\ldots,p_n,p_{n+1}} h^j_{p_1,\ldots,p_n,p_{n+1}}(y^1)^{p_1}\ldots(y^n)^{p_n}e^{ip_{n+1}t},$$

where the numbers $p_1, \ldots, p_n$ are nonnegative integers and p_{n+1} is an integer.

Again we consider the vectors

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \quad \mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}_0^n.$$

Definition 2.2.2. The monomial

$$h^j_{p_1,\ldots,p_n,p_{n+1}}(y^1)\cdots(y^n)^{p_n}e^{ip_{n+1}t}, |\mathbf{p}| \geq 2$$

is called *resonant* if the equation

$$\lambda_j = \langle \boldsymbol{\lambda}, \mathbf{p} \rangle + i p_{n+1}. \tag{2.31}$$

is satisfied. Formula (2.31) defines a *resonance* of order $p = |\mathbf{p}|$.

Definition 2.2.3. We say that system (2.27) is written in *Poincaré normal form* if the expansions of the nonlinear terms of the vector function $\mathbf{h}(\mathbf{y}, t)$ into mixed Maclaurin-Fourier series contain only resonant monomials.

As in the autonomous case, these definitions are equivalent and we have the *Poincaré-Dulac theorem*.

Theorem 2.2.2. *With the aid of the formal transformation (2.26), system (2.23) can be reduced to normal form (2.27).*

Remark 2.2.1. It should be noted that, generally speaking, *every* homogeneous vector function $\mathbf{Y}_m(\mathbf{y}, t)$ that is part of the normalizing transformation (2.26) represents a formal Fourier series. Nonetheless, since in the case considered the "parametric perturbation" contains only one basic frequency $\omega = 1$, at *each* concrete m–th step—by violation of the so-called homological equation [7] "in t"—problems of small denominators, on which we will dwell at length below, don't arise and it is possible to show that every form $\mathbf{Y}_m(\mathbf{y}, t)$ rather regularly depends on t. But with respect to all sets of variables the transformation (2.26) will, of course, be only formal.

Furthermore, for the periodic case considered, the lemma on separation of motions is likewise true.

Lemma 2.2.1. *The linear change of variables*

$$\mathbf{y} = \exp(\mathbf{D}t)\mathbf{z} \tag{2.32}$$

reduces the normalized system of equations (2.27) to the system

$$\dot{\mathbf{z}} = \mathbf{h}(\mathbf{z}, 0). \tag{2.33}$$

Proof. This lemma is actually a consequence of Lemma 1.5.8. Since (2.29) generates a generalized symmetry group, so (2.33) is an analogue of Eq. (1.129).

The substitution (2.32) allows us to get rid of not just the linear terms from the diagonalized part, but also of explicit dependence on time. Consequently, in the given situation, techniques of searching for asymptotic solutions of a semi-quasihomogeneous system are again applied formally. However, as a rule and as must be expected, the normalizing transformation (2.26) and (2.31) diverges in the presence of resonances [7]. But, as in the autonomous case, we assess the presence of asymptotic solutions of the original system (2.23) by the presence of asymptotic solutions of the quasihomogeneous truncated system (2.33).

The following theorem generalizes results from the paper [61]:

Theorem 2.2.3. *Let all the eigenvalues of the matrix* $\mathbf{A}$ *be pure imaginary or zero and let system (2.33) be positive semi-quasihomogeneous with respect to the structure given by a real matrix* $\mathbf{G}$*, whose eigenvalues have positive real parts. Consider the model system that is the quasihomogeneous truncation of (2.33) with respect to the structure generated by* $\mathbf{G}$*:*

$$\dot{\mathbf{z}} = \overset{\circ}{\mathbf{h}}_{\mathrm{q}}(\mathbf{z}). \tag{2.34}$$

Let there further exist vectors $\mathbf{z}_0^\gamma \in \mathbb{R}^n$, $\mathbf{z}_0^\gamma \neq 0$ *and a number* $\gamma = \pm 1$ *satisfying the algebraic system of equations*

$$-\gamma \mathbf{G}\mathbf{z}_0^\gamma = \mathring{\mathbf{h}}_{\mathrm{q}}(\mathbf{z}_0^\gamma). \tag{2.35}$$

Then the system of differential equations (2.23) has an asymptotic solution $\mathbf{x}$ *such that* $\mathbf{x}(t) \to 0$ *as* $t \to \gamma \times \infty$.

Proof. This is practically a word-for-word repetition of the proof of Theorem 2.1.2.

By way of corollaries, it is easy to obtain a theorem on instability.

Theorem 2.2.4. *Let all multipliers of the linear approximation for the* 2π*–periodic system (2.23) lie on the unit circle and suppose that the normal form of (2.23) is such that the model system obtained from it by discarding the diagonalized linear terms and the zeroing of time is positive semi-quasihomogeneous with respect to the structure given by the matrix* $\mathbf{G}$*, whose eigenvalues have only positive real parts. If, for the vector field of the truncation* $\mathring{\mathbf{h}}_{\mathrm{q}}$*, there is a nonzero eigenvector* $\mathbf{z}_0^-$ *such that*

$$\mathbf{G}\mathbf{z}_0^- = \mathring{\mathbf{h}}_{\mathrm{q}}(\mathbf{z}_0^-)$$

(i.e. $\gamma = -1$*), then the critical point* $\mathbf{x} = \mathbf{0}$ *of system (2.23) is unstable.*

In many practical problems, systems of equations with periodic coefficients are obtained as systems for perturbed motion in the neighborhood of a periodic trajectory of an autonomous system. Therefore Theorem 2.2.4 is essentially a theorem on the instability of periodic solutions.

Concerning the problem of searching for a suitable group that gives the quasi-homogeneous structure, the periodic case is a bit more complicated than the autonomous case. In this case too it is reasonable that we might select a model system (2.34) with the help of the Newton manifold technique and then by getting a diagonal matrix to combine with the matrices of the linear infinitesimal generators of the symmetry group for system (2.33). However, in the most general case the normal form doesn't provide information about the existence of any linear symmetry fields.

We now consider the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{f}(\mathbf{0}, t) = \mathbf{0}, \tag{2.36}$$

whose right sides depend quasiperiodically on the time.

A system of type (2.36) is more conveniently considered as an autonomous case in some $(n + d)$–dimensional extended phase space $\mathbb{R}^n \times \mathbb{T}^d$, where

$$\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z} = \{\boldsymbol{\varphi} = (\varphi^1, \ldots, \varphi^d) \bmod 2\pi\}$$

is a d–dimensional torus, $d \geq 2$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\varphi}), \quad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}, \tag{2.37}$$

where $\boldsymbol{\omega} = (\omega^1, \ldots, \omega^d) \in \mathbb{R}^d$ is the d–dimensional vector of frequencies and where $\mathbf{f} \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{T}^d)$, $\mathbf{f}(\mathbf{0}, \boldsymbol{\varphi}) \equiv \mathbf{0}$.

In the sequel we will assume that the frequencies of circling the torus are rationally independent, i.e. that no resonance relation of the form

$$\langle \boldsymbol{\omega}, \mathbf{s} \rangle = 0 \tag{2.38}$$

is satisfied for vectors $\mathbf{s} \in \mathbb{Q}^d$, $|\mathbf{s}| > 0$.

We will investigate systems of equations that are still more general than (2.37). On the manifold $\mathbb{R}^n \times \mathbb{T}^d$ we consider the system of equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\varphi}), \quad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega} + \boldsymbol{\xi}(\mathbf{x}, \boldsymbol{\varphi}), \quad \mathbf{f}, \boldsymbol{\xi} \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{T}^d), \tag{2.39}$$

for which the motion on the invariant manifold $\{\mathbf{0}\} \times \mathbb{T}^d$ represents circling the torus: $\mathbf{x} = \mathbf{0}$, $\boldsymbol{\varphi} = \boldsymbol{\omega} t + \boldsymbol{\varphi}_0$, i.e. $\mathbf{f}(\mathbf{0}, \boldsymbol{\varphi}) \equiv \mathbf{0}$, $\boldsymbol{\xi}(\mathbf{0}, \boldsymbol{\varphi}) \equiv \mathbf{0}$.

We will be forced to impose more stringent smoothness conditions on the vector functions $\mathbf{f}$, $\boldsymbol{\xi}$ and, in fact, assume that these vector functions are analytic in $\boldsymbol{\varphi}$ over some complex neighborhood of the torus $\mathbb{T}^d$.

In Eq. (2.39) we select the first nontrivial terms of the expansion in $\mathbf{x}$:

$$\dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\varphi})\mathbf{x}, \quad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}, \tag{2.40}$$

where $\mathbf{A}(\boldsymbol{\varphi}) = d_{\mathbf{x}}\mathbf{f}(\mathbf{0}, \boldsymbol{\varphi})$.

If resonances (2.38) are absent, then the system (2.33) is equivalent to an n–dimensional linear system with coefficients that are quasiperiodic in the time t. This system is sharply distinguished in its properties from analogous systems with periodic coefficients. A system with periodic coefficients gets reduced, in accordance with Floquet-Lyapunov, to autonomous form. For quasiperiodic systems comparable results simply don't exist and sufficient conditions for the reducibility of such systems are rather complicated. Apart from natural Diophantine conditions of strict incommensurability for the frequencies $\omega^1, \ldots, \omega^d$, these conditions as a rule contain certain rather harsh restrictions on the spectrum of the system considered [91], whose fulfillment is impossible to verify without integrating the system.

Another approach to this problem consists in regarding quasiperiodic systems as small perturbations of autonomous ones [19, 92]. In particular, it is shown in the article [92] that, when certain auxiliary conditions of nonresonance type on the eigenvalues of the unperturbed system and the winding frequencies are fulfilled, then the system is reducible if a small parameter belongs to a certain Cantor set of measure zero. In the article [184], the case is considered where the elements of the matrix $\mathbf{A}$ are small and, under conditions of strict nonresonance, the reducibility of (2.40) to a system with exponentially small nonconstant part is demonstrated. In order to avoid all these complications, we suppose a priori that the matrix of the linear approximation is constant: $\mathbf{A}(\boldsymbol{\varphi}) \equiv \mathbf{A}$.

We introduce resonance concepts for the problems considered. Let the numbers $\lambda_1, \ldots, \lambda_n$ be the roots of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{E}) = \mathbf{0}$.

Definition 2.2.4. A linear relationship between components of the vectors λ and ω of the form

$$\lambda_j = \langle \boldsymbol{\lambda}, \mathbf{p} \rangle + \langle \boldsymbol{\omega}, \mathbf{s} \rangle, \tag{2.41}$$

$p \in N_0^n$, $s \in Z^d$, is called a *resonance of the first kind* of order $p = |p|$, and a relation

$$\langle \boldsymbol{\lambda}, \mathbf{p} \rangle + \langle \boldsymbol{\omega}, \mathbf{s} \rangle = 0 \tag{2.42}$$

is called a *resonance of the second kind* of order $p = |p|$.

Lemma 2.2.2. *Let resonances of the second kind and of first order be absent from the system considered. Using a formal substitution*

$$\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi} + \mathbf{C}(\boldsymbol{\varphi})\mathbf{x} \bmod 2\pi, \tag{2.43}$$

where the $d \times n$ matrix $\mathbf{C}(\boldsymbol{\varphi})$ is represented as a formal multiple Fourier series, we can obtain

$$d_{\mathbf{x}}\boldsymbol{\xi}(\mathbf{0}, \boldsymbol{\varphi}) = \mathbf{0}.$$

If in addition we have satisfaction of the Diophantine condition

$$|\lambda_j + i\langle \boldsymbol{\omega}, \mathbf{s} \rangle| \geq C|\mathbf{s}|^{-a} \tag{2.44}$$

for $|\mathbf{s}| > 0$, where $C > 0$, $a > 0$ are certain constants not dependent on $\mathbf{s}$, then this substitution will be analytic in some complex neighborhood of the torus $\mathbb{T}^d$.

Proof. It is easily observed that, after application of the transformation (2.43), the linear approximating system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega} + \mathbf{B}(\boldsymbol{\varphi})\mathbf{x},$$

where $\mathbf{B}(\boldsymbol{\varphi}) = d_{\mathbf{x}}\boldsymbol{\xi}(\mathbf{0}, \boldsymbol{\varphi})$, is converted into the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega} + \mathbf{B}(\boldsymbol{\varphi})\mathbf{x} - \mathbf{C}(\boldsymbol{\varphi})\mathbf{A}\mathbf{x} - d_{\boldsymbol{\varphi}}(\mathbf{C}(\boldsymbol{\varphi})\mathbf{x})\boldsymbol{\omega}.$$

Of course, the substitution (2.43) leads likewise to new terms that are nonlinear in $\mathbf{x}$, but we restrict ourselves to considering only the approximating system that is linear in $\mathbf{x}$.

Our problem will consist of choosing a matrix $\mathbf{C}(\boldsymbol{\varphi})$ in such a way that the expression $\mathbf{B}(\boldsymbol{\varphi})\mathbf{x} - \mathbf{C}(\boldsymbol{\varphi})\mathbf{A}\mathbf{x} - d_{\boldsymbol{\varphi}}(\mathbf{C}(\boldsymbol{\varphi})\mathbf{x})\boldsymbol{\omega}$ vanishes identically. Without loss of generality we will suppose that the matrix $\mathbf{A}$ is reduced to Jordan form. Moreover, as is easily seen, it suffices to carry out the proof in the case where $\mathbf{A}$ is a Jordan block on whose diagonal there is one of the roots λ_j of the characteristic equation

$\det(\mathbf{A}-\lambda\mathbf{E})=\mathbf{0}$. The elements $c_l^k(\boldsymbol{\varphi})$ of the matrix $\mathbf{C}(\boldsymbol{\varphi})$ must satisfy the following system of linear partial differential equations on the torus $\mathbb{T}^d$:

$$\begin{aligned} \langle d_{\boldsymbol{\varphi}} c_l^k(\boldsymbol{\varphi}), \boldsymbol{\omega}\rangle + \lambda_j c_l^k(\boldsymbol{\varphi}) + c_{l-1}^k(\boldsymbol{\varphi}) &= b_l^k(\boldsymbol{\varphi}), \\ l = 2,\ldots,n, \quad k &= 1,\ldots,d \\ \langle d_{\boldsymbol{\varphi}} c_l^k(\boldsymbol{\varphi}), \boldsymbol{\omega}\rangle + \lambda_j c_l^k(\boldsymbol{\varphi}) = b_l^k(\boldsymbol{\varphi}), \quad k &= 1,\ldots,d \end{aligned} \tag{2.45}$$

If this system is solved with $l=1$ for arbitrary $b_l^k(\boldsymbol{\varphi})$, then the remaining coefficients can be found analogously by induction (here over the elements of the matrix $\mathbf{B}(\boldsymbol{\varphi})$ that are denoted by $b_l^k(\boldsymbol{\varphi})$). We expand the functions $b_l^k(\boldsymbol{\varphi})$ and $c_l^k(\boldsymbol{\varphi})$ in multiple Fourier series, substitute into (2.45) and equate coefficients of like harmonics $e^{i\langle \mathbf{s},\boldsymbol{\varphi}\rangle}$. The coefficients of these functions will then be related by

$$\big(i\langle \boldsymbol{\omega}, \boldsymbol{\varphi}\rangle + \lambda_j\big)\, c_{1,\mathbf{s}}^k = b_{1,\mathbf{s}}^k. \tag{2.46}$$

Since, by the hypothesis of the lemma, resonances of the second kind and first order are absent, the Fourier coefficients $c_{1,\mathbf{s}}^k$ are uniquely determined from (2.46). We note too that the absence of these resonances likewise means that all roots λ_j of the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{E})=\mathbf{0}$ are nonzero, so that $c_{1,0}^k=(\lambda_j)^{-1}b_{1,0}^k$.

For proving the analyticity, we use a procedure that is standard in KAM theory (see e.g. [201]).

We fix some $r>0$ and consider the scale of Banach spaces $\mathfrak{A}_\sigma$, $\sigma\in(0,1]$, where $\mathfrak{A}_\sigma$ is the Banach space of functions $c(\mathbf{w})$, $c\colon \mathcal{A}_{r\sigma}^d \to \mathbb{C}$—where the set $\mathcal{A}_{r\sigma}^d$ is the d–fold Cartesian product

$$\mathcal{A}_{r\sigma}^d = \{w\in\mathbb{C}\colon e^{-r\sigma} < |w| < e^{r\sigma}\}$$

of the annulus with itself—that are holomorphic on $\mathcal{A}_{r\sigma}^d$ and continuous on the boundary of this set, with norm

$$\|c\|_\sigma = \max_{\partial\mathcal{A}_{r\sigma}^d} |c(\mathbf{w})|.$$

It is also useful to note that the functions in $\mathfrak{A}_\sigma$ admit expansions into multiple Laurent series

$$c(\mathbf{w}) = \sum_{s_1,\ldots,s_d} (w^1)^{s_1}\ldots(w^d)^{s_d},$$

whose coefficients satisfy the well-known Cauchy estimate

$$|c_{\mathbf{s}}| \le e^{-|\mathbf{s}|r\sigma}\|c\|_\sigma.$$

If $0<\sigma'<\sigma\le 1$, then $\mathfrak{A}_\sigma\subset\mathfrak{A}_{\sigma'}$ and $\|c\|_{\sigma'}\le\|c\|_\sigma$ for any function $c\in\mathfrak{A}_\sigma$.

We consider any function $b\in\mathfrak{A}_\sigma$ and the partial differential equation, obtained from the last series of Eq. (2.45) by the substitution $w^k=e^{i\varphi^k}$, $k=1,\ldots,d$ (for brevity, from now on we will omit the indices in the equations):

$$i \langle \mathbf{w} d_{\mathbf{w}} c(\mathbf{w}), \boldsymbol{\omega} \rangle + \lambda c(\mathbf{w}) = b(\mathbf{w}), \tag{2.47}$$

where λ is some complex number satisfying Diophantine conditions of the type (2.44) and $\mathbf{w} d_{\mathbf{w}} = \left(w^1 \partial / \partial w^1, \ldots, w^d \partial / \partial w^d \right)$.

Let $c(\mathbf{w})$ be some solution of Eq. (2.47). Using the Cauchy inequality and the Diophantine conditions (2.44), we estimate this solution by the norm of the space $\mathfrak{A}_{\sigma'}$ for $\sigma' < \sigma$:

$$\begin{aligned} \|c\|_{\sigma'} &\le |\lambda|^{-1} |b_0| + \textstyle\sum_{\mathbf{s} \neq 0} \frac{|b_{\mathbf{s}}|}{|\lambda + \langle \boldsymbol{\omega}, \mathbf{s} \rangle|} e^{|\mathbf{s}|\sigma'} \\ &\le \left(|\lambda|^{-1} + C \textstyle\sum_{\mathbf{s} \neq 0} |\mathbf{s}|^a e^{-|\mathbf{s}|(\sigma - \sigma')} \right) \|b\|_{\sigma'} \le \\ &\le \left(|\lambda|^{-1} + C \textstyle\int_{\mathbb{R}^d} |\mathbf{s}|^a e^{-|\mathbf{s}|(\sigma - \sigma')} \, d\mathbf{s} \right) \|b\|_{\sigma}, \end{aligned}$$

where $d\mathbf{s} = ds_1, \ldots, ds_d$.

The above indefinite integral can be computed explicitly, so that

$$\|c\|_{\sigma'} \le (|\lambda|^{-1} + CS(d-1)\Gamma(a+d)(\sigma - \sigma'^{-(a+d)}) \|b\|_{\sigma}.$$

Here $S(d-1)$ denotes the surface area of the $(d-1)$–dimensional unit sphere, and Γ is the Euler gamma function [90].

From this it follows that the solution obtained for the equation considered will likewise be analytic, but over a somewhat smaller domain. The homomorphic solvability of the remaining equations in (2.45) is similarly proved.

The lemma is proved.

For systems of type (2.39), for which the matrix of the linear approximations is constant and where the equations on the torus don't contain terms linear in $\mathbf{x}$, there likewise exists a theory of normal forms [12]. We consider the formal coordinate substitution

$$\mathbf{x} = \mathbf{y} + \sum_{m=2}^{\infty} \mathbf{Y}_m(\mathbf{y}, \boldsymbol{\psi}), \quad \boldsymbol{\varphi} = \boldsymbol{\psi} + \sum_{m=2}^{\infty} \boldsymbol{\Psi}_m(\mathbf{y}, \boldsymbol{\psi}) \bmod 2\pi. \tag{2.48}$$

The vector functions $\mathbf{Y}_m, \boldsymbol{\Psi}_m$ are homogeneous forms $\mathbf{y}$ of degree m with coefficients presented as formal multiple Fourier series.

As a result of the transformation (2.48) we get a new system of equations

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \boldsymbol{\psi}), \quad \dot{\boldsymbol{\psi}} = \boldsymbol{\omega} + \boldsymbol{\eta}(\mathbf{y}, \boldsymbol{\psi}). \tag{2.49}$$

The matrix $\mathbf{A}$ is once again presented in the form $\mathbf{A} = \mathbf{D} + \mathbf{J}$, where $\mathbf{D}$ is the corresponding diagonal matrix and $\mathbf{J}$ is block-diagonal with zero diagonal. We likewise write the vector functions $\mathbf{g}(\mathbf{y}, \boldsymbol{\psi})$ in the form

$$\mathbf{g}(\mathbf{y}, \boldsymbol{\psi}) = \mathbf{D}\mathbf{y} + \mathbf{h}(\mathbf{y}, \boldsymbol{\psi}), \quad \mathbf{h}(\mathbf{y}, \boldsymbol{\psi}) = \mathbf{J}\mathbf{y} + \ldots,$$

where the dots signify the totality of terms nonlinear in $\mathbf{y}$.

Definition 2.2.5. We say that system (2.49) is written in *Poincaré normal form* if, for any $(z, \boldsymbol{\theta}) \in \mathbb{R}^n \times \mathbb{T}^d$, $\sigma \in \mathbb{R}$, the system of formal equations

$$\begin{aligned} &\exp(-\mathbf{D}\sigma)\mathbf{h}\,(\exp(\mathbf{D}\sigma)\mathbf{z}, \boldsymbol{\theta} + \boldsymbol{\omega}\sigma) = \\ &= \mathbf{h}(\mathbf{z}, \boldsymbol{\theta}), \boldsymbol{\eta}\,(\exp(\mathbf{D}\sigma)\mathbf{z}, \boldsymbol{\theta} + \boldsymbol{\omega}\sigma) = \boldsymbol{\eta}(\mathbf{z}, \boldsymbol{\theta}) \end{aligned} \tag{2.50}$$

holds.

This definition is equivalent to requiring that system (2.49) have a symmetry group whose infinitesimal generator yields the system of equations

$$\frac{d\mathbf{y}}{d\sigma} = \mathbf{D}\mathbf{y}, \quad \frac{d\boldsymbol{\psi}}{d\sigma} = \boldsymbol{\omega}, \tag{2.51}$$

which immediately leads to a system of differential equations in the vector functions $h, \boldsymbol{\eta}$:

$$d_{\boldsymbol{\psi}}\mathbf{h}(\mathbf{y}, \boldsymbol{\psi})\boldsymbol{\omega} = [\boldsymbol{h}(\mathbf{y}, \boldsymbol{\psi})\mathbf{D}\mathbf{y}]\,, \; d_{\mathbf{y}}\boldsymbol{\eta}(\mathbf{y}, \boldsymbol{\psi})\mathbf{D}\mathbf{y} + d_{\boldsymbol{\psi}}\boldsymbol{\eta}(\mathbf{y}, \boldsymbol{\psi})\boldsymbol{\omega} = \mathbf{0}. \tag{2.52}$$

In system (2.52) the symbol $[\cdot\,,\cdot]$ denotes the Lie algebra commutator of vector fields on $\mathbb{R}^n$; vector-matrix objects of various dimensions are multiplied by the customary rules of matrix algebra.

System (2.51) generates a group of transformations of the manifold $\mathbb{R}^n \times \mathbb{T}^d$, which acts on $\mathbb{R}^n$ like a rotation group and on $\mathbb{T}^d$ like a group of translations.

The use of Eqs. (2.50) or (2.52) for computing the coefficients of the normal form is rather difficult. By considering resonances we can get a picture of how the right sides of the normal system (2.49) should look. So let $\mathbf{D} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ and consider the mixed Fourier-Maclaurin expansions of the respective j–th and l–th components of the vector functions $\mathbf{h}$ and $\boldsymbol{\eta}$:

$$\begin{aligned} h^j &= \sum\nolimits_{p_1,\ldots,p_n,s_1,\ldots,s_d} h^j_{p_1,\ldots,p_n,s_1,\ldots,s_d} (y^1)^{p_1} \ldots (y^n)^{p_n} e^{i(\psi^1 s_1 + \ldots + \psi^d s_d)}, \\ \eta^l &= \sum\nolimits_{p_1,\ldots,p_n,\, s_1,\ldots,s_d} \eta^l_{p_1,\ldots,p_n,s_1,\ldots,s_d} (y^1)^{p_1} \ldots (y^n)^{p_n} e^{i(\psi^1 s_1 + \ldots + \psi^d s_d)}. \end{aligned}$$

Definition 2.2.6. The monomial

$$h^j_{p_1,\ldots,p_n,s_1,\ldots,s_d} (y^1)^{p_1} \ldots (y^n)^{p_n} e^{i(\psi^1 s_1 + \ldots + \psi^d s_d)}$$

is called *resonant* if the vectors $\mathbf{p}$ and $\mathbf{s}$ satisfy the relation (2.41), i.e. if a resonance of the first kind occurs. On the other hand, the monomial

$$\eta^l_{p_1,\ldots,p_n,s_1,\ldots,s_d} (y^1)^{p_1} \ldots (y^n)^{p_n} e^{i(\psi^1 s_1 + \ldots + \psi^d s_d)}$$

will be called *resonant* if $\mathbf{p}$ and $\mathbf{s}$ satisfy the relation (2.42), i.e. if a resonance of the second kind occurs.

Definition 2.2.7. We say that system (2.49) is written in *normal form* if the expansions of the vector functions $\mathbf{h}(\mathbf{y}, \boldsymbol{\psi})$, $\boldsymbol{\eta}(\mathbf{y}, \boldsymbol{\psi})$ in mixed multiple Maclaurin-Fourier series contain only resonant monomials.

An analogue of the Poincaré-Dulac theorem holds:

Theorem 2.2.5. *With the aid of the formal transformation (2.48), the system (2.39) can be reduced to the normal form (2.49).*

Theorem 2.2.5 is proved in [12] for systems of the form (2.37). However, under the assumptions made above concerning absence of resonances of the second kind and of first order, the theorem can be extended to the more general systems of form (2.39), whereby the proof will essentially repeat the one given in [12].

In contrast to systems with periodic coefficients, the systems we consider possess an additional unpleasant property. Normalization of Eq. (2.39) is a procedure of recursive calculation of homogeneous vector forms $\mathbf{Y}_m(\mathbf{y}, \boldsymbol{\psi})$, $\boldsymbol{\Psi}_m(\mathbf{y}, \boldsymbol{\psi})$, whose coefficients, without imposition of any additional requirements, are *formal* multiple Fourier series with respect to the variables $(\psi^1, \dots, \psi^d)$, where these series may already begin to diverge after the first normalization steps, i.e. for small m. In order to avoid this unpleasantness, it is necessary to impose Diophantine conditions of type (2.44) *on those pairs of vectors* $\mathbf{p}$, $\mathbf{s}$, *for which resonance doesn't occur*. We now formulate more precisely the requirements needed for this.

Lemma 2.2.3. *Suppose there exist constants $C_{M_*} > 0$, $a_{M_*} > 0$, not depending on the vector $\mathbf{s}$ and such that for those vectors $\mathbf{p}, \mathbf{s}$, $|\mathbf{p}| \le M_*$, $|\mathbf{s}| > 0$ for which resonances of types (2.41) and (2.42) don't occur, the inequalities*

$$|\lambda_j - \langle \boldsymbol{\lambda}, \mathbf{p} \rangle - i\,\langle \boldsymbol{\omega}, \mathbf{s} \rangle| \ge C_{M_*} |\mathbf{s}|^{-a_{M_*}}, \tag{2.53}$$

$$|\langle \boldsymbol{\lambda}, \mathbf{p} \rangle + i\,\langle \boldsymbol{\omega}, \mathbf{s} \rangle| \ge C_{M_*} |\mathbf{s}|^{-a_{M_*}}. \tag{2.54}$$

are valid. Then, for $m \le M_*$ the coefficients of the vector form $\mathbf{Y}_m(\mathbf{y}, \boldsymbol{\psi})$, $\boldsymbol{\Psi}_m(\mathbf{y}, \boldsymbol{\psi})$ will be analytic in $\boldsymbol{\psi}$ in some complex neighborhood of the torus $\mathbb{T}^d$.

The proof is like the proof of Lemma 2.2.2.

Of course, it is necessary to take into account that the dimensions of the complex neighborhood of the torus—in which the coefficients of a certain m–th form of the normalizing transformation are analytic—decreases as m increases. Therefore we can hardly expect that in a more or less general situation all forms in the expansion (2.48) will have coefficients that are analytic in ψ.

We formulate a lemma on the separation of motions.

Lemma 2.2.4. *Under the effect of the change of variables*

$$\mathbf{y} = \exp(\mathbf{D}t)\mathbf{z}, \quad \boldsymbol{\psi} = \boldsymbol{\omega}t + \boldsymbol{\theta} \tag{2.55}$$

the normalized system of equations (2.49) assumes the form

$$\dot{\mathbf{z}} = \mathbf{h}(\mathbf{z}, \boldsymbol{\theta}), \quad \dot{\boldsymbol{\theta}} = \boldsymbol{\eta}(\mathbf{z}, \boldsymbol{\theta}). \tag{2.56}$$

Recall that $d_{\mathbf{z}}\mathbf{h}(\mathbf{0}, \theta)$ doesn't depend on θ and is a nilpotent operator; $\eta(\mathbf{0}, \theta) = \mathbf{0}$, $d_{\mathbf{z}}\mathbf{h}(\mathbf{0}, \theta) = \mathbf{0}$.

A system of type (2.56) is a rather interesting object in itself, independent of its being obtained from the normal form of system (2.39). Consider (2.56) from the "quasihomogeneous theoretical" point of view. In the phase space $\mathbb{R}^n \times \mathbb{T}^d$ of this system we need to introduce some scale that allows us to distinguish truncations. We use the approach from Sect. 1.5 of Chap. 1. On the $\mathbb{R}^n$ "level", corresponding to the variable $\mathbf{z}$, we assign a scale corresponding to an extended group of quasihomogeneous dilations:

$$\mathbf{z} \mapsto \exp(-\mathbf{G}\sigma)\mathbf{z}.$$

The simplest transformation group that doesn't violate the topology of the torus $\mathbb{T}^d$ is the group of translations

$$\theta \mapsto \theta + \boldsymbol{\nu}\sigma, \quad \boldsymbol{\nu} \in \mathbb{R}^d.$$

If here time transforms according to the law

$$t \mapsto e^{\sigma} t,$$

then the infinitesimal generator of the given transformation group gives rise to the system of differential equations

$$\frac{d\mathbf{z}}{d\sigma} = -\mathbf{G}\mathbf{z}, \quad \frac{d\theta}{d\sigma} = \boldsymbol{\nu}, \quad \frac{dt}{d\sigma} = t. \tag{2.57}$$

We will assume that the vector functions $\mathbf{h}$, $\boldsymbol{\eta}$ depend on θ in the following way:

$$\mathbf{h} = \mathbf{h}(\mathbf{z}, \boldsymbol{\phi}), \quad \boldsymbol{\eta} = \boldsymbol{\eta}(\mathbf{z}, \boldsymbol{\phi}),$$

where $\boldsymbol{\phi} = \mathbf{P}_{\mathbf{s}}\theta \bmod 2\pi$ are the so-called resonance phases, $\mathbf{P}_{\mathbf{s}}$ is a $d' \times d$ matrix ($d' \leq d$), whose rows form a lineally independent system of vectors $\mathbf{s}$ with integral entries such that the harmonics $e^{i\langle \mathbf{s}, \theta \rangle}$ occur on the right sides of Eq. (2.56). This assumption is warranted since, in the process of normalization, the number of Fourier harmonics "diminishes" greatly.

Lemma 2.2.5. *Let $d' < d$. Then there exists a nonzero vector $\boldsymbol{\nu}$ such that the vector functions $\mathbf{h}$, $\boldsymbol{\eta}$ are invariant with respect to the family of translations $\theta \mapsto \theta + \boldsymbol{\nu}\sigma$.*

As an example, we carry out the proof for the vector function $\mathbf{h}$. For invariance, satisfaction of the following equation is necessary and sufficient:

$$\mathbf{0} = \frac{d}{d\sigma}\left(\mathbf{h}\left(\mathbf{z}, \mathbf{P}_{\mathbf{s}}(\theta + \boldsymbol{\nu}\sigma)\right)\right) = d_{\phi}\mathbf{h}\left(\mathbf{z}, \mathbf{P}_{\mathbf{s}}(\theta + \boldsymbol{\nu}\sigma)\right)\mathbf{P}_{\mathbf{s}}\boldsymbol{\nu}.$$

Since $d' < d$, there exists a nonzero vector $\boldsymbol{\nu}$ that is perpendicular to all vectors that compose the matrix $\mathbf{P_s}$, i.e. $\mathbf{P_s}\boldsymbol{\nu} = \mathbf{0}$.

The lemma is proved.

Definition 2.2.8. System (2.56) is called *quasihomogeneous*, and we will assign to its right sides the index " q " if (2.56) has an generalized symmetry group generated by the flow of system (2.57).

From Lemma 2.2.5 it follows that, if (2.56) is quasihomogeneous by some fixed matrix $\mathbf{G}$ and $\boldsymbol{\nu} = 0$, then this system will be quasihomogeneous for arbitrary $\boldsymbol{\nu} \in \operatorname{Ker} \mathbf{P_s}$.

Definition 2.2.9. System (2.56) is called *semi-quasihomogeneous* if (2.56) possesses an exponentially-asymptotic generalized symmetry group generated by the flow of system (2.57).

We consider the truncated system of equations

$$\dot{\mathbf{z}} = \mathbf{h}_{\mathrm{q}}(\mathbf{z}, \boldsymbol{\theta}), \quad \dot{\boldsymbol{\theta}} = \boldsymbol{\eta}_{\mathrm{q}}(\mathbf{z}, \boldsymbol{\theta}). \tag{2.58}$$

Particular solutions of (2.58), lying on the orbits of the group of symmetries considered, have the form

$$\mathbf{z}^{\gamma}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{z}_0^{\gamma}, \quad \boldsymbol{\theta}^{\gamma}(t) = \boldsymbol{\theta}_0^{\gamma} + \boldsymbol{\nu}\,\ln(\gamma t). \tag{2.59}$$

It should be noted that, throughout, the resonances of the phase $\phi^{\gamma}(t) = \mathbf{P_s}\boldsymbol{\theta}^{\gamma}(t)$ remain constant.

So that the system (2.58) can have a particular solution of form (2.59), $\mathbf{z}_0^{\gamma} \in \mathbb{R}^n$, $\mathbf{z}_0^{\gamma} \neq 0$, and $\boldsymbol{\theta}_0^{\gamma} \in \mathbb{T}^d$ must satisfy the following system of algebraic equations:

$$-\gamma \mathbf{G}\mathbf{z}_0^{\gamma} = \mathbf{h}_{\mathrm{q}}(\mathbf{z}_0^{\gamma}, \boldsymbol{\theta}_0^{\gamma}), \quad \boldsymbol{\nu} = \boldsymbol{\eta}_{\mathrm{q}}(\mathbf{z}_0^{\gamma}, \boldsymbol{\theta}_0^{\gamma}). \tag{2.60}$$

System (2.60) contains $n + d$ equations with $n + d$ unknowns if, besides $d' < d$, there are an additional $d - d'$ parameters entering the system. It is clear that—among all possibilities—these parameters can be selected so that system (2.60) has the required solutions.

We look at some examples that illustrate the concepts we have introduced.

Example 2.2.1. The system of three differential equations

$$\dot{z} = -\frac{1}{2}z^3, \quad \dot{\theta}^1 = z^2\cos\phi, \quad \dot{\theta}^2 = -z^2\sin\phi,$$

where $\phi = \theta^1 + \theta^2$ is the resonance phase, is quasihomogeneous with respect to the group of transformations that give the phase flow of the system

$$\frac{dz}{d\sigma} = -\frac{1}{2}z, \quad \frac{d\theta^1}{d\sigma} = \nu, \quad \frac{d\theta^2}{d\sigma} = -\nu$$

and has the family of particular solutions

$$z = -\frac{1}{\sqrt{t}},\ \theta^1 = \theta_0^1 + \frac{\sqrt{t}}{2}\ln t,\ \theta^2 = \theta_0^2 - \frac{\sqrt{t}}{2}\ln t,\ \theta_0^1 + \theta_0^2 = \frac{\pi}{4},$$

lying on the orbits of this group, with $\nu = 1/\sqrt{2}$.

If the system (2.56) is obtained from the normal form (2.49), then the appearance of the truncation strongly depends on which resonances occur. If the original system appears as (2.37), then the vector function $\boldsymbol{\eta}$ is identically equal to zero and any truncation has the form

$$\dot{\mathbf{z}} = \mathbf{h}_{\mathrm{q}}(\mathbf{z}, \boldsymbol{\theta}), \quad \dot{\boldsymbol{\theta}} = \mathbf{0},$$

and the system of equations (2.60) becomes

$$-\gamma \mathbf{G}\mathbf{z}_0^\gamma = \mathbf{h}_{\mathrm{q}}(\mathbf{z}_0^\gamma, \boldsymbol{\theta}_0^\gamma),$$

where $\boldsymbol{\theta}_0^\gamma$ should be regarded as a parameter.

An analogous situation occurs when there are no resonances of the second kind up to a sufficiently high order. In this case the expansion of $\boldsymbol{\eta}$ will begin with rather high powers of $\mathbf{z}$. But if, conversely, there are no resonances of the first kind of low orders, then the truncations will have the appearance of the equations:

$$\dot{\mathbf{z}} = \mathbf{0}, \quad \dot{\boldsymbol{\theta}} = \boldsymbol{\eta}_{\mathrm{q}}(\mathbf{z}, \boldsymbol{\theta}),$$

which, of course, *cannot have* asymptotic solutions.

Truncations of (2.58) will, as before, be called *model*.

Particular solutions of system (2.58) of type (2.59) generate solutions of the full system (2.56) with analogous asymptotic. More precisely, we have

Theorem 2.2.6. *Suppose that the right sides of system (2.56) are smooth functions on $\mathbb{R}^n \times \mathbb{T}^d$, that system (2.56) is semi-quasihomogeneous with respect to the group of transformations that generate the phase flow (2.57), and that there exist $\mathbf{z}_0^\gamma \in \mathbb{R}^n$, $\mathbf{z}_0^\gamma \neq \mathbf{0}$, and $\boldsymbol{\theta}_0^\gamma \in \mathbb{T}^d$ satisfying (2.60) for $\gamma = \pm 1$, $\boldsymbol{\nu} \in \operatorname{Ker} \mathbf{P_s}$. Then system (2.56) has a particular solution with smooth components with asymptotic*

$$\mathbf{z}(t) \sim (\gamma t)^{-\mathbf{G}}\mathbf{z}_0^\gamma, \quad \boldsymbol{\theta}(t) \sim \boldsymbol{\theta}_0^\gamma + \boldsymbol{\nu}\ln(\gamma t)$$

as $t^\chi \to \gamma \times \infty$.

The proof follows from Theorem 1.5.2. We only need note that formal particular solutions of the full system can be written in the form of the series

$$\begin{aligned} \mathbf{z}(t) &= (\gamma t)^{-\mathbf{G}} \textstyle\sum_{k=0}^{\infty} \mathbf{z}_k\, (\ln(\gamma t))\, (\gamma t)^{-k\beta}, \\ \boldsymbol{\theta}(t) &= \boldsymbol{\nu}\ln(\gamma t) + \textstyle\sum_{k=0}^{\infty} \theta_k\, (\ln(\gamma t))\, (\gamma t)^{-k\beta}, \end{aligned} \tag{2.61}$$

where $\mathbf{z}_k$, θ_k are polynomial vector functions, whereby the zero order coefficients of the series (2.61) are constants.

If the system (2.56) is obtained from the normal form (2.49) of system (2.39), then the formal particular solutions (2.49) will have the appearance:

$$\mathbf{y}(t) = \exp\left(\mathbf{D}t - \mathbf{G}\ln(\gamma t)\right) \sum_{k=0}^{\infty} \mathbf{z}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta},$$
$$\boldsymbol{\psi}(t) = \boldsymbol{\omega} t + \boldsymbol{\nu}(\ln(\gamma t) + \sum_{k=0}^{\infty} \boldsymbol{\theta}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta}.$$

If all eigenvalues of the matrix $\mathbf{A}$ lie on the imaginary axis then, in order that $\mathbf{y}(t)$ converge formally to zero, it is necessary that β be positive, i.e that the system (2.56) be positive semi-quasihomogeneous and that all the eigenvalues of the matrix $\mathbf{G}$ have positive real part. We introduce conditions for the existence of "non formal" asymptotic solutions.

Theorem 2.2.7. *Let the right sides of (2.39) satisfy the following conditions:*

(a) The system (2.40) is reduced and the matrix of the reduced system has only pure imaginary or zero eigenvalues;
(b) In the system there are no first order resonances of the second kind and there exist $C > 0$, $a > 0$ *such that the inequality (2.44) holds;*
(c) The normal form (2.49) is such that the system (2.56) obtained from it by rejection of corresponding terms is positive semi-quasihomogeneous, and all eigenvalues of the matrix $\mathbf{G}$ *that gives the quasihomogeneous structure are positive;*
(d) For selecting the quasihomogeneous truncation (2.58) it is required that we compute M *forms of the normalizing transformation (2.48), and that for some large* $M_* > M$ *there exist constants* $C_{M_*} > 0$, $a_{M_*} > 0$ *such that the vectors* $\mathbf{p}$, $\mathbf{s}$, *where* $|\mathbf{p}| \leq M_*$, $|\mathbf{s}| > 0$, *for which the resonance relations (2.41) and (2.42) don't hold, satisfy the Diophantine conditions (2.53) and (2.54);*
(e) There exist $\mathbf{z}_0^\gamma \in \mathbb{R}^n$, $\mathbf{z}_0^\gamma \neq \mathbf{0}$, $\boldsymbol{\theta}_0^\gamma \in \mathbb{T}^d$ *satisfying (2.60) for some* $\gamma = \pm 1$, $\boldsymbol{\nu} \in \operatorname{Ker} \mathbf{P}_\mathbf{s}$.

Then system (2.39) has a particular asymptotic solution $(\mathbf{x}(t), \boldsymbol{\varphi}(t)) \to \{\mathbf{0}\} \times \mathbb{T}^d$ *as* $t \to \gamma \times \infty$.

The proof is based on ideas already used in the proof of Theorem 2.1.2. We give only an outline, while commenting on the hypothesis for the theorem.

Conditions (a) and (b) are needed so as to be able to apply a normalizing transformation to the system considered. After a formally complete normalization and substitution (2.55), we get a semi-quasihomogeneous system, which by conditions (c) and (e) has a formal solution with the required asymptotic. We carry out a partial normalization up to order M_* and then make the substitution (2.55). The fact that the partial normalization will be an analytic transformation is guaranteed by condition (d). After perturbing the particular solution (2.59) of the truncated system (2.58)

$$\mathbf{z}(t) = (\gamma t)^{-\mathbf{G}} \left(\mathbf{z}_0^\gamma + \mathbf{u}(\gamma t)\right), \quad \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0^\gamma + \boldsymbol{\nu}\ \ln(\gamma t) + \mathbf{v}(\gamma t),$$

and introducing logarithmic time $\tau = \ln(\gamma t)$, we reduce the solution of the problem to applying the lemmas of Sect. 1.3 of Chap. 1. The Kovalevsky matrix will be autonomous in this case, so that Lemmas 1.2.2 and 1.2.3 are applicable without further hypothesis.

The application of these lemmas thus completes the proof of Theorem 2.2.7.

By the form of the particular solution (2.59) of the truncated system (2.58), it is clear that the limit motion toward which our solution tends does not in general represent the standard winding about the torus. Roughly speaking, this motion will be the composition of two windings, the first of which proceeds in real time and the second in logarithmic. We likewise formulate an instability theorem.

Theorem 2.2.8. *Let all the hypothesis of the preceding theorem be fulfilled with $\gamma = -1$ in condition (e). Then the invariant manifold $\{\mathbf{0}\} \times \mathbb{T}^d$ of system (2.39) is unstable.*

The proof follows from the existence of a solution that is asymptotic to $\{\mathbf{0}\} \times \mathbb{T}^d$ as $t \to -\infty$.

In conclusion we make some remarks about the properties of a particular solution of the form (2.59) on the torus:

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\nu} \ln t, \quad \mathbf{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_n) \in \mathbb{T}^n, \quad \boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n. \tag{2.62}$$

We recall that the motion $t \mapsto \mathbf{v}(t)$ is said to be *uniformly distributed* (Weyl), if for any Jordan measurable domain $D \subset \mathbb{T}^n$

$$\lim_{T\to\infty} \frac{\mu_D(T)}{T} = \frac{\text{meas}\, D}{\text{meas}\, \mathbb{T}^n},$$

where $\mu_D(T)$ is the total path length on the interval $[O, T]$, where $\mathbf{v}(t) \in D$, where meas D is the measure of the domain D and, in particular, meas $\mathbb{T}^n = (2\pi)^n$. Weyl gave the following criteria for uniform distribution:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{i <\mathbf{m},\mathbf{v}(t)}\, dt = 0 \tag{2.63}$$

for all integral vectors $\mathbf{m} = (m_1, \ldots, m_n) \neq 0$. In part, conditionally periodic motions

$$\mathbf{v} = \nu t + \mathbf{v}_0$$

with a nonresonant set of frequencies ν are uniformly distributed on $\mathbb{T}^n$ (*Bolya-Sierpinski-Weyl theorem*).

However, the motion (2.62) does not satisfy condition (2.63). Already for $n = 1$:

$$\frac{1}{\tau} \int_0^\tau e^{i \ln t}\, dt = \frac{e^{i \ln \tau}}{1+i} + \frac{\text{const}}{\tau},$$

which oscillates and doesn't tend to zero as $\tau \to \infty$.

Definition 2.2.10. Let $t \mapsto \lambda(t)$ be a positive continuous function, where

$$\int_0^\infty \lambda(t)\,dt = \infty.$$

The motion $t \mapsto \mathbf{v}(t)$ is said to be (R,λ) *uniformly distributed* if for any Jordan measurable domain $D \subset \mathbb{T}^n$

$$\lim_{\tau\to\infty} \int_0^\tau \lambda(t) f(\mathbf{v}(t))\,dt \;/ \int_0^\tau \lambda(t)\,dt = \frac{\text{meas}\, D}{\text{meas}\, \mathbb{T}^n},$$

where f is the characteristic function of the domain D.

The integral on the left is the mean weighted time interval, over which the point $\mathbf{v}(t)$ is located in the domain D. If $\lambda(t) = 1$, then we get a determination of Weyl's uniform distribution.

In the paper [113] it is proved that, if the function $t\lambda(t)$ is monotone for $t \ge t_0$ and

$$\tau\lambda(\tau) \;/ \int_0^\tau \lambda(t)\,dt \to 0,$$

then the motion (2.62) with an incommensurable set of frequencies $\nu_1, \dots, \nu_n$ will be (R,λ) uniformly distributed on $\mathbb{T}^n$.

It is clear that these conditions are satisfied by the functions $\lambda(t) = \frac{1}{t}$ and $\lambda(t) = (t \ln^\gamma t)^{-1}$, for all $0 \le \gamma \le 1$.

2.3 Hamiltonian Systems

We will apply the method developed to the search for asymptotic solutions of Hamiltonian systems of differential equations in critical cases. For simplicity, we limit ourselves to the autonomous situation and will assume that the system is written in Darboux canonical coordinates [6, 9]. We thus consider a system of differential equations in the even-dimensional linear space $\mathbb{R}^{2n}$:

$$\dot{\mathbf{x}} = \mathbf{I}\,dH(\mathbf{x}), \tag{2.64}$$

where $\mathbf{x} = (\mathbf{u},\mathbf{v}) = (u_1,\dots,u_n,v_1,\dots,v_n)$, the *Hamiltonian* H is some smooth function and the matrix $\mathbf{I}$, the so-called symplectic unit matrix, has the form

$$\mathbf{I} = \begin{pmatrix} \mathbf{0} & -\mathbf{E} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}.$$

Although conceptually $\mathbf{x}$ is a vector, here we will relinquish the contravariant notation for the components and will henceforth use only the lower indices. Let the origin $\mathbf{x} = \mathbf{0}$ be an equilibrium point, i.e. $dH(\mathbf{0}) = \mathbf{0}$. For a Hamiltonian system, stability (and hence also instability) in the future and in the past are equivalent [100, 171]. A consequence of this general fact is the known theorem: the characteristic polynomial of a linear Hamiltonian system only contains even degrees of the argument. Stability is possible only in critical cases, i.e. when all eigenvalues of the linear approximation system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \mathbf{I}\, d^2 H(\mathbf{0})\mathbf{x} \tag{2.65}$$

are pure imaginary or zero: $\lambda_{j,n+j} = \pm i\omega_j$, $\omega_j \geq 0$, $j = 1, \ldots, n$. In the sequel we consider only these critical cases.

For studying problems of existence of asymptotic solutions of the Hamiltonian system (2.64), it is possible to use the scenario laid out in Sect. 2.1 of the present chapter. The first step is the reduction of the system to Poincaré normal form. In this, the normalizing transformation will in general be noncanonical and the equations obtained will lose their Hamiltonian form. It is, however, intuitively clear that the conditions for the existence of asymptotic solutions of a system of equations having Hamiltonian form should have a simpler appearance than in the general case. This idea is corroborated by results that have been obtained on the existence of asymptotic solutions of Hamiltonian systems with two degrees of freedom for *principal* resonances [10, 11, 56, 135, 136]. In particular, in Hamiltonian systems difficulties don't arise that are connected with algebraic insolubility, at least not at low levels of degeneracy. It is therefore natural to try to develop an analogous procedure that accounts for the Hamiltonian structure of the original equations.

Consider the expansion of a Hamiltonian function $H(\mathbf{x})$ into a Maclaurin series in the neighborhood of the equilibrium point $\mathbf{x} = \mathbf{0}$. Without loss of generality we can assume that $H(\mathbf{0}) = 0$ and that the expansion begins with quadratic terms:

$$H(\mathbf{x}) = H_2(\mathbf{x}) + \ldots .$$

The dots denote the totality of terms whose orders are greater than or equal to three.

With the help of a real nonsingular linear symplectic transformation, the quadratic part $H_2(\mathbf{x})$ of the Hamiltonian can be reduced to a simpler form, the so-called normal form. We will assume that this quadratic normalization has already been carried out.

If the matrix $\mathbf{A}$ of the linearized system is diagonalized, then H_2 is reduced to the form:

$$H_2 = \frac{1}{2}\sum_{j=1}^{n} \sigma_j(\omega_j^2 u_j^2 + v_j^2), \quad \sigma_j = \pm 1. \tag{2.66}$$

Now let the matrix **A** be similar to a pair of Jordan blocks of odd order $n = 2k + 1$ with pure imaginary eigenvalues $\pm i\omega$. Then H_2 can be reduced to the form:

$$H_2 = \Big[\tfrac{\sigma}{2}\sum\nolimits_{j=1}^{k}(\omega^2 u_{2j}u_{2k-2j+2} + v_{2j}v_{2k-2j+2}) - \\ -\sum\nolimits_{j=1}^{k}(\omega^2 u_{2j-1}u_{2k-2j+3} + v_{2j}v_{2k-2j+3})\Big] - \sum\nolimits_{j=1}^{2k} u_j v_{j+1}. \tag{2.67}$$

Here and in the sequel we set $\sigma = \pm 1$.

If the order of the Jordan blocks is even ($n = 2k$), then the normal form of H_2 will appear as follows:

$$H_2 = \tfrac{\sigma}{2}\Big[\sum\nolimits_{j=1}^{k}(\omega^{-2}v_{2j-1}v_{2k-2j+1} + v_{2j}v_{2k-2j+2}) - \\ -\sum\nolimits_{j=1}^{k}(\omega^2 u_{2j+1}u_{2k-2j+1} + u_{2j+2}u_{2k-2j+2})\Big] - \\ -\omega^2\sum\nolimits_{j=1}^{k} u_{2j-1}v_{2j} + \sum\nolimits_{j=1}^{k} u_{2j}v_{2j-1}. \tag{2.68}$$

In the sequel we will be especially interested in the case $k = 1$ (two degrees of freedom), where

$$H_2 = \frac{\sigma}{2}(\omega^{-2}v_1^2 + v_2^2) - \omega^2 u_1 v_2 + u_2 v_1. \tag{2.69}$$

For pairs of Jordan blocks of odd order $n = 2k + 1$ with zero eigenvalues, the normal form has the appearance:

$$H_2 = \sum_{j=1}^{2k} u_j v_{j+1}. \tag{2.70}$$

It is clear that $H_2 \equiv 0$ if $k = 0$.

Finally, if **A** is similar to *one* Jordan block of order $2k = 2n$ with eigenvalue zero, then the quadratic part of the Hamiltonian reduces to:

$$H_2 = \frac{\sigma}{2}\left[\sum_{j=1}^{k-1} u_j u_{k-j} - \sum_{j=1}^{k} v_j u_{k-j+1}\right] - \sum_{j=1}^{k-1} u_j v_{j+1}. \tag{2.71}$$

If $k = 1$, then

$$H_2 = \frac{\sigma}{2}v_1^2. \tag{2.72}$$

For two degrees of freedom ($k = 2$) this normal form has the appearance:

$$H_2 = \frac{\sigma}{2}\left[u_1^2 - 2v_1v_2\right] - u_1v_2. \tag{2.73}$$

It should be noted that the Hamiltonians corresponding to $\sigma = +1$ and $\sigma = -1$, respectively, cannot be transformed into one another.

The general situation is described in the well-known theorem of Williamson [195]:

Theorem 2.3.1. *Let all roots of the linear approximating system (2.65) be pure imaginary or zero. Then either the Hamiltonian reduces to the form (2.66) or else the symplectic phase space of the system decomposes as the direct sum of skew-orthogonal real symplectic subspaces such that H_2 is represented as the sum of forms of types (2.66)–(2.73) on these subspaces.*

Let $\mathfrak{g}$ be the Lie algebra of infinitely differentiable Hamiltonian functions on $\mathbb{R}^{2n}$. The Poisson bracket $\{\cdot\,,\cdot\}$ on $\mathfrak{g}$ is given in the standard way:

$$\{F, G\} = \langle \mathbf{I}\, dF, dG\rangle = \langle d_u F, d_v G\rangle - \langle d_v F, d_u G\rangle.$$

There exists a connection between the Poisson bracket and the commutator of vector fields on $\mathbb{R}^{2n}$:

$$[\mathbf{I}\, dF, \mathbf{I}\, dG] = \mathbf{I}\, d\{F, G\}. \tag{2.74}$$

We consider the $n(2n+1)$–dimensional subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$, whose coordinates are quadratic in some fixed coordinate system. From the well-known theorem of Cartan on replicas (see e.g. [7]), it follows that the quadratic form H_2 can be uniquely represented as the sum of two forms:

$$H_2(\mathbf{x}) = H_2'(\mathbf{x}) + H_2''(\mathbf{x})$$

that commute with one another:

$$\{H_2', H_2''\} = 0,$$

where H_2', H_2'' are, respectively, semi-simple and nilpotent elements of the subalgebra $\mathfrak{h}$.

This means that the matrix $\mathbf{I}\, d^2 H_2'$ is diagonalizable and that $\mathbf{I}\, d^2 H_2''$ is nilpotent. Therefore, without loss of generality, we can assume that the quadratic form H_2' has the appearance (2.66).

We now consider some canonical transformations $\mathbf{x} = (\mathbf{u}, \mathbf{v}) \mapsto\mapsto \mathbf{y} = (\boldsymbol{\xi}, \boldsymbol{\eta})$ that preserve the linear part of the system (2.64). Let the generating function $S = S(\mathbf{v}, \boldsymbol{\xi})$ of this transformation be developed in a formal Maclaurin series

$$S(\mathbf{v},\boldsymbol{\xi}) = \langle \mathbf{v},\boldsymbol{\xi}\rangle + \sum_{m=2}^{\infty} S_m(\mathbf{v},\boldsymbol{\xi}), \tag{2.75}$$

where the $S_m(\mathbf{v},\boldsymbol{\xi})$ are homogeneous polynomials of degree m in the "old" coordinates $\mathbf{v}$ and in the "new" momenta $\boldsymbol{\xi}$.

The connection between the "old" and the "new" variables is determined by the relations:

$$\begin{aligned} \mathbf{u} &= d_{\mathbf{v}} S(\mathbf{v},\boldsymbol{\xi}) = \boldsymbol{\xi} + \sum_{m=2}^{\infty} d_{\mathbf{v}} S_m(\mathbf{v},\boldsymbol{\xi}), \\ \boldsymbol{\eta} &= d_{\boldsymbol{\xi}} S(\mathbf{v},\boldsymbol{\xi}) = \mathbf{v} + \sum_{m=2}^{\infty} d_{\boldsymbol{\xi}} S_m(\mathbf{v},\boldsymbol{\xi}). \end{aligned} \tag{2.76}$$

which, unfortunately, aren't explicit. In order to obtain an explicit dependency it is necessary, for example, to extract $\boldsymbol{\xi}$ from (2.76) as a function of the variables $\mathbf{u}$ and $\mathbf{v}$, and to substitute the result into the second relation.

The "new" Hamiltonian function $G(\mathbf{y}) = G(\boldsymbol{\xi},\boldsymbol{\eta})$ will satisfy the relation

$$H(d_{\mathbf{v}} S(\mathbf{v},\boldsymbol{\xi}),\mathbf{v}) = G(\boldsymbol{\xi}, d_{\boldsymbol{\xi}} S(\mathbf{v},\boldsymbol{\xi})).$$

Just like $H(\mathbf{x})$, the Hamiltonian $G(\mathbf{y})$ has the form

$$G(\mathbf{y}) = G_2(\mathbf{y}) + \ldots,$$

where

$$G_2(\mathbf{y}) = H_2(\mathbf{y}).$$

We write the Hamiltonian function $G(\mathbf{y})$ in the form

$$G(\mathbf{y}) = G_2'(\mathbf{y}) + F(\mathbf{y}),$$

where $G_2'(\mathbf{y}) = H_2'(\mathbf{y})$ denotes the "semi-simple component" of the quadratic part of $G(\mathbf{y})$.

Definition 2.3.1. We say that a Hamiltonian system of differential equations

$$\dot{\mathbf{y}} = \mathbf{I}\, dG(\mathbf{y}) \tag{2.77}$$

is reduced to *Birkhoff normal form* if the functions G_2' and F commute, i.e. if

$$\left\{G_2'(\mathbf{y}), F(\mathbf{y})\right\} \equiv 0. \tag{2.78}$$

This definition means that (2.77) has a symmetry group that generates the linear Hamiltonian system

$$\frac{d\mathbf{y}}{d\sigma} = \mathbf{I}\, dG_2'(\mathbf{y}).$$

From the existence of a homomorphism of the Lie algebra of Hamiltonian functions to the Lie algebra of vector fields it follows that, if the relation (2.74) reduces the system (2.77) to Birkhoff normal form, then the system has Poincaré normal form.

Definition 2.3.1 likewise admits an interpretation in the language of resonances. We consider for simplicity the case where the first approximation to system (2.77) isn't degenerate:

$$\det(d^2G(\mathbf{0})) \neq 0.$$

Using a linear symplectic transformation we pass to complex canonical coordinates:

$$\xi_j = \frac{1}{\sqrt{2\omega_j}}(i\alpha_j - \beta_j), \quad \eta_j = \sqrt{\frac{\omega_j}{2}}(\alpha_j - i\beta_j), \quad j = 1, \dots, n,$$

in which G_2' has the following form:

$$G_2' = -i\sum_{j=1}^{n} \sigma_j \omega_j \alpha_j \beta_j.$$

We write the Maclaurin expansion of the function $F = G - G_2'$

$$F = \sum_{p_1,\dots,p_n,s_1,\dots,s_n} F_{p_1,\dots,p_n,s_1,\dots,s_n} \alpha_1^{p_1} \dots \alpha_n^{p_n} \beta_1^{s_1} \dots \beta_n^{s_n}$$

and consider the three vectors:

$$\boldsymbol{\omega} = (\sigma_1\omega_1, \dots, \sigma_n\omega_n) \in \mathbb{R}^n, \mathbf{p} = (p_1, \dots, p_n) \in \mathbb{N}_0^n,$$
$$\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{N}_0^n.$$

Definition 2.3.2. The monomial

$$F_{p_1,\dots,p_n,s_1,\dots,s_n} \alpha_1^{p_1} \dots \alpha_n^{p_n} \beta_1^{s_1} \dots \beta_n^{s_n}, \quad |\mathbf{p}| + |\mathbf{s}| \geq 3$$

is called *resonant* if

$$\langle \omega_\sigma, \mathbf{p} - \mathbf{s} \rangle = 0. \tag{2.79}$$

Moreover, relation (2.79) is called a *resonance* of order $r = |\mathbf{p}| + |\mathbf{s}|$.

Definition 2.3.3. We say that system (2.77) is written in *Birkhoff normal form* if the expansion of the function $F - G_2''$ in a Maclaurin series contains only resonant monomials.

This definition also extends to the case where the system of first approximation (2.65) has some null frequencies. In this case, under the substitution $(\boldsymbol{\xi}, \boldsymbol{\eta}) \mapsto (\boldsymbol{\alpha}, \boldsymbol{\beta})$, the variables corresponding to null frequencies don't get transformed.

A yet more transparent view of the Birkhoff normal form is given by the following construction. From canonical Cartesian coordinates $\boldsymbol{\xi}, \boldsymbol{\eta}$ we pass to polar coordinates $\boldsymbol{\rho}, \boldsymbol{\theta}$:

$$\xi_j = \sqrt{\frac{2\rho_j}{\omega_j}} \cos\theta_j, \quad \eta_j = \sqrt{2\omega_j\rho_j}\sin\theta_j, \quad j = 1, \ldots, n.$$

In canonical polar coordinates the quadratic form G_2' depends only on the momenta $\boldsymbol{\rho}$:

$$G_2' = \sum_{j=1}^{n} \sigma_j \omega_j \rho_j,$$

and the resonance relations (2.79) can be rewritten in the following way:

$$\langle \boldsymbol{\omega}_\sigma, \mathbf{r} \rangle = 0, \tag{2.80}$$

$\mathbf{r} \in \mathbb{Z}^n$, where $r = |\mathbf{r}|$ is the order of the resonance.

It is obvious that the number of independent resonances in relation (2.80) can't exceed $n-1$. Let $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(n)} \in \mathbb{Z}^n$, $d \le n-1$, be some "maximal" set of linearly independent vectors such that

$$\langle \boldsymbol{\omega}_\sigma, \mathbf{r}^{(l)} \rangle = 0, \quad l = 1, \ldots, d.$$

We introduce the resonant phases:

$$\phi_l = \langle \mathbf{r}^{(l)}, \boldsymbol{\theta} \rangle \bmod 2\pi, \quad l = 1, \ldots, d.$$

In the new variables the function $F^* = F - G_2''$ depends on $(\boldsymbol{\rho},\boldsymbol{\theta}) \in \mathbb{R}^n \times \mathbb{T}^n$. It can be shown (see e.g. [100]) that, if the system is reduced to Birkhoff normal form, then it is possible to choose a system of vectors $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(d)}$ such that

$$F^* = F^*(\boldsymbol{\rho}, \boldsymbol{\phi}), \;\; \boldsymbol{\phi} = (\phi_1, \ldots, \phi_d), \\ (\boldsymbol{\rho}, \boldsymbol{\phi}) \in \mathbb{R}^n \times \mathbb{T}^d, \;\; d \le n-1,$$

i.e. dependency on angular variables appears only in the resonant phases. In all this it should be kept in mind that the variables $\boldsymbol{\rho}, \boldsymbol{\phi}$ are not canonical. From this fact we can make the following useful observation (see e.g. [100]). Let the matrix of the linearized system (2.65) be reduced to diagonal form, i.e. $G_2''(\mathbf{y}) \equiv 0$. We extend the linearly independent system of vectors $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(d)}$ to a basis for the space $\mathbb{R}^n$ by appending linearly independent vectors $\mathbf{r}^{(d+1)}, \ldots, \mathbf{r}^{(n)}$, where for any $l = 1, \ldots, d$ and $k = d+1, \ldots, n$ we have

$$\langle \mathbf{r}^{(l)}, \mathbf{r}^{(k)} \rangle = 0.$$

Then we can write $n - d$ independent quadratic formal integrals of system (2.77):

$$G_2^{(k)} = \langle \mathbf{r}^{(k)}, \boldsymbol{\rho} \rangle = \sum\nolimits_{j=1}^{n} \mathbf{r}^{(k)} \rho_j = \\ = \tfrac{1}{2} \sum\nolimits_{j=1}^{n} \mathbf{r}_j^{(k)} (\omega_j^2 \xi_j^2 + \eta_j^2), \quad k = d+1, \dots, n. \tag{2.81}$$

We formulate a theorem on reducibility to normal form.

Theorem 2.3.2 (Birkhoff's theorem). *Let all roots of the system of first approximation (2.65) be pure imaginary or zero. Then the Hamiltonian system (2.64) can be reduced to Birkhoff normal form (2.77).*

A theory of normal forms of Hamiltonian systems of equations likewise exists for cases where not all roots of the characteristic equation of the system of first approximation lie on the imaginary axis. However, we will not consider these situations, since in this chapter we are interested only in critical cases. The idea of an algorithm for reduction to normal form that uses an implicit canonical transformation with generating function (2.75) is due to Birkhoff [17]. There also exist methods for realizing the construction of the normalizing transformation explicitly, the so-called Deprit-Hori method [45, 82]. An extensive survey of problems connected with the normalization procedure for Hamiltonian systems can be found in [43]. As in the case of reduction to Poincaré normal form, the reduction of system (2.64) to Birkhoff normal form (2.77) is as a rule divergent; in particular, the series (2.75) that represents the generating function of the normalizing transformation is divergent. The reason for divergence here, and also for the transformation to Poincaré normal form, is the presence of anomalously small denominators. The fundamental result on divergence was established by Siegel [172] (see also [112]). As in the general case, a lemma on separation of motions holds for Hamiltonian systems written in normal form. We introduce the notation:

$$\mathbf{D} = \mathbf{I}\, d G_2'(\mathbf{0}).$$

Lemma 2.3.1. *The linear canonical change of variables*

$$\mathbf{y} = \exp(\mathbf{Dt})\mathbf{z} \tag{2.82}$$

takes the normalized system of equations (2.77) to the system

$$\dot{\mathbf{z}} = \mathbf{I}\, \mathbf{dF}(\mathbf{z}). \tag{2.83}$$

Since a Hamiltonian system written in Birkhoff normal form also has Poincaré normal form, this lemma is a consequence of Lemma 2.1.1.

Further simplifications are possible in system (2.83) that are related to the so-called *generalized* Birkhoff-Gustavson normal form and are analogous to the Birkhoff normal form [13] for general systems of differential equations (see e.g. the paper [48]). However, for our concrete goals of obtaining existence conditions for asymptotic solutions, the use of the procedure of reduction to Birkhoff normal form in its classical version is completely adequate.

So, from now on the fundamental object of our investigations will be system (2.83), to which it is possible to apply the technique of selecting a quasihomogeneous truncation as described in the preceding chapter. This technique in the Hamiltonian case that we now consider has some peculiarities. The difficulty is that not every group of quasihomogeneous dilations

$$\mathbf{z} \mapsto \mu^{\mathbf{G}}\mathbf{z}, \quad t \mapsto \mu^{-1}t \tag{2.84}$$

preserves the Hamiltonian nature of the equations.

It seems completely natural that studying the behavior of the Hamiltonian function $F(\mathbf{z})$ under the action of the transformation $\mathbf{z} \mapsto \mu^{\mathbf{G}}\mathbf{z}$ should provide a basis for the investigation.

Definition 2.3.4. We say that the function $F(\mathbf{z})$ is *quasihomogeneous* of degree $Q \in \mathbb{R}$ with respect to the structure generated by the matrix $\mathbf{G}$, and we use the notation $F_Q(\mathbf{z})$, if

$$F_Q(\mu^{\mathbf{G}}\mathbf{z}) = \mu^{\mathbf{Q}} F_Q(\mathbf{z}). \tag{2.85}$$

If we abstract from the fact that the matrix $\mathbf{G}$ *uniquely* determines the one-parameter group (2.84) of transformations of extended phase space, then the homogeneity degree of any specific function is determined to within a multiplicative constant. The multiplication of elements of the matrix $\mathbf{G}$ by some real number a leads to the multiplication of Q by the same number.

Definition 2.3.5. We say that the function $F(\mathbf{z})$ is *semi-quasihomogeneous* with respect to the structure generated by the matrix $\mathbf{G}$ if it can be represented as a formal sum

$$F(\mathbf{z}) = \sum_{m=0} F_{Q+\chi m}(\mathbf{z}) \tag{2.86}$$

such that, for some $\beta \in \mathbb{R} \setminus \{0\}$, we have the relation:

$$F(\mu^{\mathbf{G}}\mathbf{z}) = \sum_{m=0} \mu^{Q+\beta m} F_{Q+\chi m}(\mathbf{z}). \tag{2.87}$$

If $\beta > 0$, then we say that the function $F(\mathbf{z})$ is *positive semi-quasihomogeneous*; but if $\beta < 0$, then we speak of a *negative semi-quasihomogeneous* function $F(\mathbf{z})$. We call the number $\chi = \pm 1$ the *sign* of semi-quasihomogeneity. It is clear that the $F_{Q+\chi m}$ are themselves quasihomogeneous functions, of higher “degree” when $\chi = +1$ and lower when $\chi = -1$.

Thus semi-quasihomogeneous functions are functions represented as a sum

$$F(\mathbf{z}) = F_Q(\mathbf{z}) + F^*(\mathbf{z}),$$

where $F_Q(\mathbf{z})$ is a quasihomogeneous truncation and $F^*(\mathbf{z})$ is a “remainder”, in some sense not affecting its local properties.

The concepts of quasihomogeneity and semi-quasihomogeneity for functions, as for vector fields, can likewise be treated from the point of view of Newton diagrams and polytopes [32].

Let the matrix $\mathbf{G}$ be diagonal: $\mathbf{G} = \text{diag}(g_1, \ldots, g_N)$ (in our case $N = 2n$) and let the function $F(\mathbf{z})$ be represented in the form of a Maclaurin series

$$F(\mathbf{z}) = \sum_{i_1,\ldots,i_N} F_{i_1,\ldots,i_N} z_1^{i_1} \ldots z_N^{i_N}, \tag{2.88}$$

where $i_1, \ldots, i_N$ are nonnegative integers.

Definition 2.3.6. Let $F_{i_1,\ldots,i_N} z_1^{i_1} \ldots z_N^{i_N}$ be any nontrivial monomial in the expansion (2.88). In the space $\mathbb{R}^N$ we distinguish the geometric point with coordinates $(i_1, \ldots, i_N)$ and call the totality of all such points the *Newton diagram* $\mathfrak{D}$ of the function $F(\mathbf{z})$ and its convex hull the *Newton polytope* $\mathfrak{P}$.

It is clear that the exponents of monomials of the quasihomogeneous truncation lie in the hyperplane π_Q given by the equation

$$g_1 i_1 + \ldots g_N i_N = Q. \tag{2.89}$$

This hyperplane "supports" the Newton polytope $\mathfrak{P}$ in the case of positive semi-quasihomogeneity. That is if Γ is a face of $\mathfrak{P}$ that lies in π_Q, then for any $(i_1, \ldots, i_N) \in \mathfrak{P} \setminus \Gamma$ the inequality

$$g_1 i_1 + \ldots g_N i_N > Q \tag{2.90}$$

is satisfied.

But, in the case of negative semi-quasihomogeneity, π_Q "covers" the Newton polytope $\mathfrak{P}$, i.e.

$$g_1 i_1 + \ldots g_N i_N < Q \tag{2.91}$$

for any $(i_1, \ldots, i_N) \in \mathfrak{P} \setminus \Gamma$.

Let the elements of the matrix $\mathbf{G}$ be rational nonnegative numbers. It is clear that every function represented in the form of a series (2.88) can be expanded as a series (2.86) in quasihomogeneous forms, i.e. an arbitrary function $F(\mathbf{z})$ of the indicated type is positive semi-quasihomogeneous. The algorithm for such an expansion is based on the geometric interpretation of inequalities (2.90) and (2.91). We consider a family of hyperplanes π_Q given by inequalities (2.89) under continuous variation of the parameter Q from zero to infinity. For some $Q = Q^*$ this hyperplane is tangent to the polytope $\mathfrak{P}$. Points of the Newton diagram $\mathfrak{D}$ that fall on the given hyperplane determine a quasihomogeneous truncation $F_{Q^*}(\mathbf{z})$. By further increase in the parameter, the points of $\mathfrak{D}$ will fall on π_Q only for discrete values of Q, the distances between which will be multiples of some positive rational number β. The points of the Newton diagram that fall on π_Q for fixed $Q > Q^*$ will give the higher quasihomogeneous forms in the expansion (2.86). If $F(\mathbf{z})$ is a polynomial, i.e. if the Newton diagram $\mathfrak{D}$ contains but a finite number of points, then $F(\mathbf{z})$ may as well be considered a negative semi-quasihomogeneous function, following the above analytical procedure, but we now let Q decrease from some value $Q_{\max}$ to zero.

It can be shown in a completely natural way that the gradient of a quasihomogeneous (resp. semi-quasihomogeneous) function is itself a quasihomogeneous (resp. semi-quasihomogeneous) vector field. It is easily seen, however, that this isn't true in the general case. We will find conditions that are sufficient for a Hamiltonian system of equations with quasihomogeneous Hamiltonian to be quasihomogeneous.

We have the following

Lemma 2.3.2. *Under the action of transformation (2.84) the gradient of a quasi-homogeneous function is transformed as follows:*

$$dF_Q(\mu^{\mathbf{G}}\mathbf{z}) = \mu^{-G^T+QE}\, dF_Q(\mathbf{z}), \tag{2.92}$$

where the symbol $(\cdot)^T$ *denotes transposition.*

Proof. We consider Eq. (2.85) and make a small change $\Delta\mathbf{z}$ in the variable $\mathbf{z}$:

$$F_Q\left(\mu^{\mathbf{G}}(\mathbf{z}+\Delta\mathbf{z})\right) = F_Q\left(\mu^{\mathbf{G}}(\mathbf{z})\right) + \left(dF_Q\left(\mu^{\mathbf{G}}(\mathbf{z})\right), \mu^{\mathbf{G}}\Delta\mathbf{z}\right) + \\ + o\left(\|\Delta\mathbf{z}\|\right) = \mu^Q\left(F_Q(\mathbf{z}) + \left(dF_Q(\mathbf{z}), \Delta\mathbf{z}\right)\right) + o\left(\|\Delta\mathbf{z}\|\right).$$

Equating coefficients of terms that are linear in $\Delta\mathbf{z}$, we get Eq. (2.92).

The lemma is proved.

We further limit ourselves to consideration of the case where the quasihomogeneous group given by a matrix $\mathbf{G}$ independently transforms coordinates and momenta. For the components of the variable $\mathbf{z}$ we retain for simplicity the previous notation $(\boldsymbol{\xi}, \boldsymbol{\eta})$, used earlier for the components of $\mathbf{y}$. We set

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_\xi & 0 \\ 0 & \mathbf{G}_\eta \end{pmatrix}.$$

Lemma 2.3.3. *If the matrix equation*

$$\mathbf{G}_{\boldsymbol{\xi}} + \mathbf{G}_{\boldsymbol{\eta}}^{\mathbf{T}} = (Q-1)\mathbf{E}, \tag{2.93}$$

holds, then system (2.83) is quasihomogeneous.

The proof follows from Lemma 2.65. In fact,

$$d_\xi F_Q\left(\mu^{\mathbf{G}_\xi}\boldsymbol{\xi}, \mu^{\mathbf{G}_\eta}\boldsymbol{\eta}\right) = \mu^{-\mathbf{G}_\xi^T+Q\mathbf{E}}\, d_\xi F_Q(\boldsymbol{\xi}, \boldsymbol{\eta}),$$
$$d_\eta F_Q\left(\mu^{\mathbf{G}_\xi}\boldsymbol{\xi}, \mu^{\mathbf{G}_\eta}\boldsymbol{\eta}\right) = \mu^{-\mathbf{G}_\eta^T+Q\mathbf{E}}\, d_\eta F_Q(\boldsymbol{\xi}, \boldsymbol{\eta}).$$

Therefore, if Eq. (2.93) is satisfied, the Hamiltonian system of equations with Hamiltonian F_Q will be quasihomogeneous.

The lemma is proved.

It is clear that a Hamiltonian system with *semi*-quasihomogeneous Hamiltonian will be *semi*-quasihomogeneous, provided that (2.93) is satisfied.

In choosing truncations it is necessary to take into consideration the following circumstance. The matrices $\mathbf{G}_\xi$ and $\mathbf{G}_\eta$ must satisfy Eq. (2.93), which contains the quantity Q as a parameter. In this way, we also fix *in advance* the *degree* of the quasihomogeneous truncation, which may be chosen, for example, by analysis of the Newton polytope of the Hamiltonian $F(\mathbf{z})$. But if the truncation obtained has some other degree, then we must consider another pair of matrices $\mathbf{G}_\xi$ and $\mathbf{G}_\eta$.

As before, we consider the model system

$$\dot{\mathbf{z}} = \mathbf{I}\, dF_Q(\mathbf{z}) \tag{2.94}$$

and consider the question of existence of particular solutions of this system in the form of quasihomogeneous rays.

As was noted above in the general case, in the presence of d independent resonances system (2.83) has $n-d$ independent formal integrals (2.81). The Hamiltonian vector field of the form

$$\frac{d\mathbf{z}}{d\sigma} = \mathbf{I}\, dG_2^{(k)}, \quad k = 1, \ldots, n-d$$

generates the symmetry group of the system (2.83) considered.

Matrices for generating various quasihomogeneous structures for system (2.94) can be sought in the following form:

$$\mathbf{G}_{\boldsymbol{\delta}} = \mathbf{G} + \sum_{j=1}^{n-d} \delta_k \mathbf{D}_k, \quad \mathbf{D}_k = \mathbf{I}\, d^2 G_2^{(k)}(\mathbf{0}), \quad k = 1, \ldots, n-d. \tag{2.95}$$

Here $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{n-d})$ is a set of arbitrary parameters, $\mathbf{G}$ is some previously determined matrix with respect to which system (2.94) is quasihomogeneous, and (2.83) is semi-quasihomogeneous, commuting with each of the matrices $\mathbf{D}_k$.

For the Hamiltonian case we can formulate an analogue of Theorem 2.1.2.

Theorem 2.3.3. *Let system (2.83) be positive semi-quasi-homogeneous in the* ***Hamiltonian sense*** *with respect to the structure of some matrix* $\mathbf{G}$ *with positive eigenvalues. Starting from (2.95), we consider the auxiliary Hamiltonian*

$$F_{\boldsymbol{\delta}} = F_Q(\mathbf{z}) + \sum_{j=1}^{n-d} \boldsymbol{\delta}_k G_2^{(k)}, \tag{2.96}$$

where the matrix $\mathbf{I}\, d^2 G_2^{(k)}(\mathbf{0})$ *commutes with* $\mathbf{G}$. *Let there exist a set of parameters* $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{n-d})$*, a nonzero vector* $\mathbf{z}_0^{\gamma} \in \mathbb{R}^{2\mathbf{n}}$ *and a number* $\gamma = \pm 1$ *satisfying the algebraic system of equations*

$$-\gamma \mathbf{G}\mathbf{z}_{\mathbf{0}}^{\mathbf{fl}} = \mathbf{I}\,\mathbf{dF}_{\mathbf{œ}}(\mathbf{z}_{\mathbf{0}}^{\gamma}). \tag{2.97}$$

Then the original system (2.64) has an asymptotic solution $\mathbf{x}(t) \to \mathbf{0}$ *as* $t \to \gamma \times \infty$.

The proof is based on application of Theorem 2.1.2.
We analogously formulate an assertion on instability.

Theorem 2.3.4. *Let the linear approximating system (2.65) have only zero or pure imaginary roots and let the normalized system (2.77) be such that the system obtained from it by rejection of quadratic terms in the Hamiltonian function, corresponding to the diagonalized component of the matrix for (2.65), is positive semi-quasihomogeneous with respect to the structure given by some matrix* $\mathbf{G}$ *with positive eigenvalues. If, for some auxiliary Hamiltonian (2.96), a nonzero vector* $\mathbf{z}_{\mathbf{0}}^{-} \in \mathbb{R}^{\mathbf{2n}}$ *can be found satisfying the system of algebraic equations*

$$\mathbf{G}\mathbf{z}_{\mathbf{0}}^{-} = \mathbf{I}\,dF_{\delta}(\mathbf{z}_0^{-}), \tag{2.98}$$

then the equilibrium position $\mathbf{x} = \mathbf{0}$ *of system (2.64) is unstable.*

The proof is based on the existence of a particular asymptotic solution of system (2.64), $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$, resulting from (2.98).

If $\boldsymbol{\delta} \neq \mathbf{0}$, then the "supporting" solution of the truncated system is a twisted ray.

It is quite possible that, if (2.97) has a solution $\mathbf{z}_0^{+}$ as $\gamma = +1$, then there also exists a solution $\mathbf{z}_0^{-}$ ($\gamma = -1$) for the very same choice of parameters δ. This possibility is confirmed by all examples known to date and is consistent with the reversibility property of Hamiltonian systems.

To illustrate the method described, we consider the problem of existence of an asymptotic trajectory and the stability of the equilibrium position for a Hamiltonian system with two degrees of freedom in the case of a neutral first approximation system. We consider first the case where both parts are nonzero and the matrix of the linearized system reduces to diagonal form.

By the substitution

$$\xi_j \mapsto \frac{1}{\sqrt{\omega_j}}\xi_j, \quad \eta_j \mapsto \sqrt{\omega_j}\,\eta_j, \quad j = 1, 2,$$

the quadratic part of the normalized Hamiltonian takes on a form that is more convenient for subsequent investigation:

$$H_2 = G_2 = \frac{1}{2}\left(\sigma_1\omega_1(\xi_1^2 + \eta_1^2) + \sigma_2\omega_2(\xi_2^2 + \eta_2^2)\right).$$

If the quantities σ_1 and σ_2 have identical sign, then the Hamiltonian H can play the role of a Lyapunov function, from which follows the stability of the equilibrium position considered. Ignoring this trivial case we can henceforth assume without loss of generality that $\sigma_1 = +1$ and $\sigma_2 = -1$.

The well-known Arnold-Moser theorem [3, 141] asserts that if the system is free of resonances through the fourth degree, i.e. if

$$r_1\omega_1 - r_2\omega_2 \neq 0, \quad r_1, r_2 \in \mathbb{N}, \quad r_1 + r_2 \leq 4,$$

then *as a rule* stability occurs in the system and asymptotic solutions are absent.

Here the expression "as a rule" means that for stability there must still be satisfied some inequality on the coefficients of the fourth degree in the expansion of the Hamiltonian function that guarantees that, in the space of Hamiltonian systems where the frequencies of the small vibrations are not resonant, unstable systems have "measure zero". Interesting additions to this theorem can be found in the paper of Bryuno [31].

Thus we consider the problem of the stability of the equilibrium position and the existence of asymptotic solutions of Hamiltonian systems of equations when resonances are present among the frequencies. The fundamental results in this direction can be found in the papers [10, 11, 56, 134–136, 175–178]. We will add some different interpretations to these results and supplement them with some new ones. We first note that, in two-dimensional space, the vector with coordinates (ω_1, ω_2) is the "unique" vector perpendicular to $\mathbf{r} = (r_1, r_2)$. Therefore G_2 is, within a nonzero factor, the unique nontrivial quadratic integral of the normalized system, and the auxiliary Hamiltonian (2.95) figuring in the formulations of Theorems 2.3.3 and 2.3.4 will have the form

$$F_\delta(\mathbf{z}) = F_Q(\mathbf{z}) + \delta G_2(\mathbf{z}). \tag{2.99}$$

In the "strict semisimple" cases considered, i.e. where the matrix of the first approximation system is diagonalized, the $F_Q(\mathbf{z})$ will be homogeneous functions of the third or fourth degree.

Example 2.3.1. Resonance 1 : 2. This resonance is the simplest of all. Let $r_1 = 1$, $r_2 = 2$, $\omega_1 = 2\omega_2 = \omega$. The truncated Hamiltonian F_Q represents the totality of cubic terms in the normal form (see e.g. [134]):

$$F_3 = a\left(\eta_1(\xi_2^2 - \eta_2^2) + 2\xi_1\xi_2\eta_2\right). \tag{2.100}$$

In this and the following examples we will give the available formulas for the first nonquadratic terms of the normalized Hamiltonian, referring to the original papers and omitting computational details. We simply note that these formulas can be obtained from commutativity conditions of the unknown functions with G_2.

In the case considered the parameter δ can be set equal to zero. The Hamiltonian system of equations with Hamiltonian (2.100) *almost always* has two one-parameter families of particular linear solutions of the form:

$$\mathbf{z}^\pm(t)\left(\frac{1}{\pm t}\xi_{1_0}^\pm, \frac{1}{\pm t}\eta_{1_0}^\pm, \frac{1}{\pm t}\xi_{2_0}^\pm, \frac{1}{\pm t}\eta_{2_0}^\pm,\right),$$

where $\mathbf{z}_0^{\pm} = \left(\xi_{1_0}^{\pm}, \eta_{1_0}^{\pm}, \xi_{2_0}^{\pm}, \eta_{2_0}^{\pm}\right)$ is an eigenvector of the Hamiltonian vector field (2.100), so that

$$\begin{aligned} \xi_{1_0}^{\pm} &= \pm\sqrt{2r}\cos 2\phi, & \eta_{1_0}^{\mp} &= \pm\sqrt{2r}\sin 2\phi, \\ \xi_{2_0}^{\pm} &= \pm 2\sqrt{r}\cos\phi, & \eta_{2_0}^{\mp} &= \pm 2\sqrt{r}\sin 2\phi, \end{aligned}$$

where ϕ is a parameter of the family and

$$r = \frac{1}{8a^2}.$$

Thus, for

$$a \neq 0,$$

the system studied has two one-parameter families of asymptotic solutions, one of which tends to the equilibrium position as $t \to +\infty$, the other as $t \to -\infty$. The existence of the latter family signals instability. Thus for the $1:2$ resonance, instability in the Hamiltonian system is typical. The consideration of the $a = 0$ case requires additional investigation.

Example 2.3.2. Resonance $1:3$. Let $r_1 = 1$, $r - 2 = 3$, $\omega_1 = 3\omega_2 = \omega$. The truncated Hamiltonian F_Q represents the totality of all terms of fourth degree in the normal form (see e.g. [100]):

$$\begin{aligned} F_4 = a\left(\xi_1\eta_2(3\xi_2^2 - \eta_2^2) + \xi_2\eta_1(\xi_2^2 - 3\eta_2^2)\right) + c_{02}(\xi_2^2 + \eta_2^2)^2 + \\ + c_{11}(\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) + c_{20}(\xi_1^2 + \eta_1^2)^2. \end{aligned} \tag{2.101}$$

We will find conditions under which the truncated Hamiltonian system with Hamiltonian (2.101) has two one-parameter families of solutions in the form of twisted rays:

$$\begin{aligned} \mathbf{z}^{\pm}(t) = \Bigg(& \frac{1}{\sqrt{\pm t}}\left(\xi_{1_0}^{\pm}\cos(3\delta\omega\ln(\pm t)) - \eta_{1_0}^{\pm}\sin(3\delta\omega\ln(\pm t))\right), \\ & \frac{1}{\sqrt{\pm t}}\left(\eta_{1_0}^{\pm}\cos(3\delta\omega\ln(\pm t)) + \xi_{1_0}^{\pm}\sin(3\delta\omega\ln(\pm t))\right), \\ & \frac{1}{\sqrt{\pm t}}\left(\xi_{2_0}^{\pm}\cos(\delta\omega\ln(\pm t)) - \eta_{2_0}^{\pm}\sin(\delta\omega\ln(\pm t))\right), \\ & \frac{1}{\sqrt{\pm t}}\left(\eta_{2_0}^{\pm}\cos(\delta\omega\ln(\pm t)) + \xi_{2_0}^{\pm}\sin(\delta\omega\ln(\pm t))\right) \Bigg). \end{aligned}$$

We set:

$$A = 3c_{02} + 3c_{11} + c_{20}, \quad B = 3\sqrt{3}a$$

and consider first the case

$$|B| > |A|. \tag{2.102}$$

The vectors $\mathbf{z}_0^{\pm} = (\xi_{1_0}^{\pm}, \eta_{1_0}^{\pm}, \xi_{2_0}^{\pm}, \eta_{1_0}^{\pm})$,

$$\xi_{1_0}^{\pm} = \sqrt{2r}\cos(\psi^{\pm} - 3\phi), \qquad \eta_{1_0}^{\pm} = \sqrt{2r}\sin(\psi^{\pm} - 3\phi),$$
$$\xi_{2_0}^{\pm} = \sqrt{6r}\cos\phi, \qquad \eta_{2_0}^{\pm} = \sqrt{6r}\sin\phi,$$

where ϕ is a parameter of the families, where

$$\psi^{\pm} = \frac{\pi}{2}(1 \mp 1) \mp \arcsin\frac{A}{B}$$

is a resonant phase, and where

$$r = \frac{1}{4\sqrt{B^2 - A^2}},$$

represents the eigenvectors of the Hamiltonian vector field with auxiliary Hamiltonian (2.99), with

$$\delta = 2r\frac{3c_{02} - c_{11} - c_{20}}{\omega}.$$

Thus, when the inequality (2.102) is satisfied, the system considered has two one-parameter families of asymptotic solutions, which indicates instability. It is known that when (2.102) is violated, i.e. when $|B| < |A|$, then the equilibrium position of the system is stable [134]. Thus, in the parameter space of the truncated system, the set of values of the parameters for which stability occurs has positive measure.

Example 2.3.3. Resonance $1:1$ in the absence of Jordan blocks. In this case $r_1 = r_2 = 1$, $\omega_1 = \omega_2 = \omega$. The truncated Hamiltonian function F_Q represents the totality of fourth degree terms in the normal form (see e.g. [175]):

$$\begin{aligned}
F_4 = {} & (\xi_1^2 + \eta_1^2)\,(a(\xi_1\xi_2 - \eta_1\eta_2) + b(\xi_1\eta_2 + \xi_2\eta_1)) + \\
& + (\xi_2^2 + \eta_2^2)\,(c(\xi_1\xi_2 - \eta_1\eta_2) + d(\xi_1\eta_2 + \xi_2\eta_1)) + \\
& + e\left((\xi_1\xi_2 - \eta_1\eta_2)^2 - (\xi_1\eta_2 + \xi_2\eta_1)^2\right) + \\
& + c_{02}(\xi_2^2 + \eta_2^2)^2 + c_{11}(\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) + c_{20}(\xi_1^2 + \eta_1^2)^2. \qquad (2.103)
\end{aligned}$$

If some additional conditions are satisfied, then the model Hamiltonian system with Hamiltonian (2.103)—and likewise in the case where the resonance is $1:3$—has two one-parameter families of solutions in the form of twisted rays:

$$\begin{aligned}
\mathbf{z}^{\pm}(t) = \Bigg(& \frac{1}{\sqrt{\pm t}}(\xi_{1_0}^{\pm}\cos(\delta^{\pm}\omega\ln(\pm t)) - \eta_{1_0}^{\pm}\sin\left(\delta^{\pm}\omega\ln(\pm t)\right), \\
& \frac{1}{\sqrt{\pm t}}(\eta_{1_0}^{\pm}\cos(\delta^{\pm}\omega\ln(\pm t)) + \xi_{1_0}^{\pm}\sin\left(\delta^{\pm}\omega\ln(\pm t)\right), \\
& \frac{1}{\sqrt{\pm t}}(\xi_{2_0}^{\pm}\cos(\delta^{\pm}\omega\ln(\pm t)) - \eta_{2_0}^{\pm}\sin\left(\delta^{\pm}\omega\ln(\pm t)\right), \\
& \frac{1}{\sqrt{\pm t}}(\eta_{2_0}^{\pm}\cos(\delta^{\pm}\omega\ln(\pm t)) + \xi_{2_0}^{\pm}\sin\left(\delta^{\pm}\omega\ln(\pm t)\right)\Bigg)
\end{aligned}$$

We consider an auxiliary function:

$$\Psi(\psi) = c_{02} + c_{11} + c_{20} + (a + c)\cos\psi + (b + d)\sin\psi + e\cos 2\psi. \quad (2.104)$$

We suppose that the function (2.104) has a real root at which the derivative of the function is nonzero. The function (2.104) is smooth and 2π-periodic, and it is easy to see that, if there exists a root ψ^+ of the equation $\Psi(\psi) = 0$ such that $\Psi\left(\psi^+\right) > 0$, then there likewise is a root ψ^- of this equation for which $\Psi'\left(\psi^-\right) > 0$.

The eigenvectors $\mathbf{z}_0^\pm = (\xi_{1_0}^\pm, \eta_{1_0}^\pm, \xi_{2_0}^\pm, \eta_{2_0}^\pm)$ of the Hamiltonian vector field with auxiliary Hamiltonian (2.99) are computed as follows:

$$\begin{aligned} \xi_{1_0}^\pm &= \sqrt{2r^\pm}\cos(\psi^\pm - \phi), & \eta_{1_0}^\pm &= \sqrt{2r^\pm}\sin(\psi^\pm - \phi), \\ \xi_{2_0}^\pm &= \sqrt{6r^\pm}\cos\phi, & \eta_{2_0}^\pm &= \sqrt{6r^\pm}\sin\phi, \end{aligned}$$

where ϕ is a parameter of the family and the resonant phases $\psi^\pm$ are roots of the function (2.104),

$$r^\pm = \pm\frac{1}{4\Psi'^\pm}).$$

The value of the parameter δ in the auxiliary Hamiltonian (2.99) is computed by the formula:

$$\delta = \delta^\pm = -2r^\pm \frac{2(c_{20} - c_{02}) + (a - c)\cos\psi^\pm + (b - d)\sin\psi^\pm}{\omega}.$$

Thus, if the function (2.104) has roots at which its derivative is nonzero, then the system considered has two one-parameter families of asymptotic solutions that tend to the equilibrium position as $t \to +\infty$ and as $t \to -\infty$, which guarantees instability. If the function (2.104) does not have real roots then, as shown in [175], we have stability by Lyapunov. As in the preceding example, stability *isn't* excluded for the resonance 1 : 1.

The example considered next doesn't at all relate to the class of "strictly semisimple" cases.

Example 2.3.4. Resonance 1: 1 in the presence of two two-dimensional Jordan Blocks. We are obliged to compute the nonlinear terms of moderate degree of the normal form of the Hamiltonian system, whose quadratic part has the form (2.69). For the solution of this problem we can use the normal form obtained in the preceding example.

The linear canonical transformation

$$\begin{aligned} u_1 &\mapsto \sqrt{\tfrac{1}{2\omega}}(u_2 + v_1), & v_1 &\mapsto \sqrt{\tfrac{\omega}{2}}(v_2 - u_1), \\ u_2 &\mapsto -\sqrt{\tfrac{\omega}{2}}(v_2 + u_1), & v_2 &\mapsto \sqrt{\tfrac{1}{2\omega}}(u_2 - v_1) \end{aligned} \quad (2.105)$$

takes the quadratic part (2.69) of the Hamiltonian of the problem to the quadratic form

$$H_2 = H_2' + H_2'', \quad H_2' = \frac{\omega}{2}\left((u_1^2+v_1^2)-(u_2^2+v_2^2)\right),$$
$$H_2'' = \frac{\sigma}{4\omega}\left((u_1-v_1)^2+(u_2-v_1)^2\right), \quad \omega_1=\omega_2=\omega.$$

We reduce the system to Birkhoff normal form $(\mathbf{u},\mathbf{v}) \mapsto (\boldsymbol{\xi}, \boldsymbol{\eta})$. The totality of terms of fourth degree of the normalized Hamiltonian, as in the preceding example, will likewise have the form (2.103). The linear canonical transformation inverse to (2.105) will give us the required normal form. The expressions for the terms of fourth degree in this case are rather clumsy and thus we won't write them here. Having disposed of the "semisimple component" of the quadratic part, we obtain the Hamiltonian systems of type (2.83).

We consider the group of quasihomogeneous dilations

$$\xi_j \mapsto \mu\xi_j, \quad \eta_j \mapsto \mu^2\eta_j, \quad j=1,2.$$

By (2.69), (2.103), and (2.105), it follows from the Hamiltonian function of a system of type (2.83) that this group "cuts" the quasihomogeneous truncation

$$F_4 = \frac{1}{2}\left(\sigma(\omega^{-2}\eta_1^2+\eta_2^2)+\frac{A}{2}(\omega\xi_1^2+\omega^{-1}\xi_2^2)^2\right), \tag{2.106}$$

where

$$A = b+d-e+c_{20}+c_{11}+c_{02}.$$

In this case the degree of homogeneity equals four.

The matrix of the transformation group that was used satisfies relation (2.93) and therefore the Hamiltonian system with Hamiltonian (2.106) will be quasihomogeneous.

The model Hamiltonian system with Hamiltonian (2.106) has two solutions in the form of quasihomogeneous rays:

$$\mathbf{z}^{\pm}(t) = \left(\frac{1}{\pm t}\xi_{1_0}^{\pm}, \frac{1}{(\pm t)^2}\eta_{1_0}^{\pm}, \frac{1}{\pm t}\xi_{2_0}^{\pm}, \frac{1}{(\pm t)^2}\eta_{2_0}^{\pm}\right),$$

where

$$\xi_{1_0}^{\pm} = \xi_{2_0}^{\pm} = \xi_0 = \left(-A\sigma(1+\omega^{-2})\right)^{-1/2},$$
$$\eta_{1_0}^{\pm} = \mp\sigma\omega^2\xi_0, \quad \eta_{2_0}^{\pm} = \mp\sigma\xi_0,$$

under the condition

$$A\sigma < 0. \tag{2.107}$$

Thus if the inequality (2.107) is satisfied, then the system under study has two asymptotic solutions that tend to the equilibrium position as $t \to +\infty$ for the one and as $t \to -\infty$ for the other, which guarantees instability. But if the inequality opposite to (2.107) holds then, as shown in [175], the system is Lyapunov stable. Consequently, in this example, instability is not the only possibility. The sufficient condition for instability (2.107) was obtained in the paper [175], starting from other considerations. Yet another nonlinear canonical transformation can be applied to the system that allows us to "kill off" a number of fourth degree terms in the normal form. This transformation is analogous to the one used for obtaining the Birkhoff-Gustavson normal form [48]. The above inequality was obtained starting just from the classical Birkhoff normal form.

We proceed to consider the case where at least one of the eigenfrequencies is zero.

Example 2.3.5. One null frequency in the absence of a Jordan block. Let $\omega_1 - \omega \neq 0$, $\omega_2 = 0$. With the aid of the transformation

$$\xi_1 \mapsto \frac{1}{\sqrt{\omega}}, \quad \eta_1 \mapsto \sqrt{\omega}\eta_1$$

the quadratic part of the Hamiltonian function of the normalized system can be reduced to the following simple form:

$$H_2 = G_2 = \frac{\omega}{2}(\xi_1^2 + \eta_1^2).$$

It can happen that some of the first nonquadratic homogeneous forms G_m, $m = 3, \ldots, M-1$ of the expansion of the Hamiltonian $G(\mathbf{y})$ turn out to be zero. Then the truncated Hamiltonian will have the form [176]:

$$F_M = \sum_{k=0}^{[M/2]} f_{M-2k}(\xi_2, \eta_2)(\xi_1^2 + \eta_1^2)^k, \tag{2.108}$$

where f_{M-2k} is a homogeneous polynomial of degree $M - 2k$ in the variables ξ_2, η_2.

The system with homogeneous Hamiltonian (2.108) clearly has an invariant manifold $\xi_1 = \eta_1 = 0$, and on this manifold the system will have the form

$$\dot{\xi}_2 = \frac{\partial f_M}{\partial \eta_2}(\xi_2, \eta_2), \quad \dot{\eta}_2 = \frac{\partial f_M}{\partial \xi_2}(\xi_2, \eta_2). \tag{2.109}$$

We will seek conditions for the existence of asymptotic solutions of system (2.109) of the form

$$\xi_2^{\pm}(t) = \frac{1}{(\pm t)^{\alpha}}\xi_{2_0}^{\pm}, \quad \eta_2^{\pm}(t) = \frac{1}{(\pm t)^{\alpha}}\eta_{2_0}^{\pm}, \quad \alpha = \frac{1}{M-2}.$$

Since the homogeneous function f_M is a first integral of system (2.109), we have

$$f_M(\xi_{2_0}^{\pm}, \eta_{2_0}^{\pm}) = 0.$$

We denote the two-dimensional vector with coordinates $(\xi^{\pm}_{2_0}, \eta^{\pm}_{2_0})$ by $\mathbf{c}^{\pm}_0 = \|\mathbf{c}^{\pm}_0\|\mathbf{e}^{\pm}_0$, where $\mathbf{e}^{\pm}_0$ is the unit vector. Then, by Euler's theorem on homogeneous functions, we have

$$M f_M(\mathbf{e}^{\pm}_0) = \langle df_M(\mathbf{e}^{\pm}_0), \mathbf{e}^{\pm}_0 \rangle = 0,$$

i.e. any vector $\mathbf{e}^{\pm}_0$ that is a zero of the function f_M is perpendicular to the gradient of this function, computed at $\mathbf{e}^{\pm}_0$.

We suppose that the set of zeros of the function f_M does not coincide with the set of zeros of its gradient, so that the form f_M is necessarily skew-symmetric. Let $\mathbf{e}^{\pm}_0$ be the unique vector such that $f_M(\mathbf{e}^{\pm}_0) = 0$ and $df_M(\mathbf{e}^{\pm}_0) \neq 0$. Then there exists a number $a \in \mathbb{R}$, $a \neq 0$ such that

$$\mathbf{I}\, df_M(\mathbf{e}^{\pm}_0) = a\mathbf{e}^{\pm}_0, \quad \mathbf{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If $a > 0$, then (2.109) has an asymptotic solution as $t \to -\infty$, and $\|\mathbf{c}^{-}_0\| = (\alpha/a)^{\alpha}$, but in the opposite case ($a < 0$) (2.109) has an asymptotic solution as $t \to +\infty$, and $\|\mathbf{c}^{+}_0\| = (\alpha/a)^{\alpha}$.

Nonetheless it is easy to show that, if there exist points on the unit circle at which the gradient of the function f_M is tangent to this circle and is directed counterclockwise, then there exist points at which this gradient is tangent to the sphere and is directed clockwise. For this it suffices to use the periodicity of the function $\Psi(\psi) = f_M(\cos\psi, \sin\psi)$ and its derivative. This shows that (2.109) has an asymptotic solutions, both for $t \to +\infty$ and for $t \to -\infty$.

Consequently the following is a condition for the existence and instability of asymptotic trajectories for the system considered: the set of zeros of the homogeneous form f_M does not intersect the set of zeros of its gradient. In the problem under consideration, at least for even M, stability likewise is not an exclusive phenomenon: as shown in the paper [176], if f_M has a fixed sign, then stability occurs by Lyapunov.

Example 2.3.6. One zero frequency in the presence of a two-dimensional Jordan block. In accordance with Theorem 2.3.1 (Williamson's theorem) the quadratic term of the Hamiltonian for the problem can be reduced to the form:

$$H_2 = \frac{\omega}{2}(u_1^2 + v_1^2) + \frac{\sigma}{2}v_2^2. \tag{2.110}$$

We reduce the system to Birkhoff normal form and again assume that some set of forms G_m, $m = 3, \ldots, M-1$ of the expanded Hamiltonian $G(\mathbf{y})$ are identically equal to zero, and that the M–th form F_M has, as in the preceding case, the appearance (2.108).

We write an expansion of the homogeneous form f_{M-2k} as a sum of elementary monomials

$$f_{M-2k}(\xi_2,\eta_2) = \sum_{l=0}^{M-2k} f_{M-2k}^{M-2k-l,i}\xi_2^{M-2k-l}\eta_2^l$$

and consider the quasihomogeneous scale (2.93):

$$\xi_j \mapsto \mu^{2\alpha}\xi_j, \quad \eta_j \mapsto \mu^{2\alpha+1}\eta_j, \quad j=1,2, \quad \alpha = \frac{1}{M-2}.$$

Using (2.108) and (2.110), it is not difficult to show that, in the case considered, this scale "cuts" from a Hamiltonian system of type (2.83), the following quasihomogeneous truncation:

$$F_Q = \frac{\sigma}{2}\eta_2^2 \sum_{k=0}^{[M/2]} f_{M-2k}^{M-2k,0}\xi_2^{M-2k}\eta_1^{2k}, \quad Q = 2\alpha M. \tag{2.111}$$

The Hamiltonian system of equations with Hamiltonian (2.111) has an invariant manifold $\xi_1 = \eta_1 = 0$ on which the truncated system assumes the form:

$$\dot{\xi}_2 = \sigma\eta_2, \quad \dot{\eta}_2 = M f_M^{M,0}\xi_2^{M-1}. \tag{2.112}$$

System (2.112) has two asymptotic solutions

$$\xi_2^{\pm}(t) = \frac{1}{(\pm t)^{2\alpha}}\xi_{2_0}^{\pm}, \quad \eta_2^{\pm}(t) = \frac{1}{(\pm t)^{2\alpha+1}}\eta_{2_0}^{\pm},$$

where

$$\xi_{2_0}^{\pm} = \xi_0 = \left(-\frac{2\alpha(2\alpha+1)\sigma}{M f_M^{M,0}}\right)^{\alpha}, \quad \eta_{2_0}^{\pm} = \pm 2\sigma\alpha\xi_0,$$

provided that one of the two following conditions is satisfied:

(a) M is odd and

$$f_M^{M,0} \neq 0,$$

(b) M is even and the inequality

$$\sigma f_M^{M,0} < 0 \tag{2.113}$$

holds.

If one of these two conditions is satisfied, then the system considered is unstable and has two asymptotic solutions, one of which converges to the equilibrium position as $t \to +\infty$, the other as $t \to -\infty$. Thus, for odd M, instability is the rule. But if M is even and the inequality opposite to (2.113) is satisfied then, as was shown by A.G. Sokol'skiy [176], the system is Lyapunov stable.

In order to find sufficient conditions for instability in the last two examples, we needn't actually resort to Birkhoff's normal form. The conditions for instability in these two problems were found by A.G. Sokol'skiy, who employed a transformation analogous to that of the Birkhoff-Gustavson extended normal form [48]. We obtain instability conditions using just the procedure of truncation of the *original* Hamiltonian and show that under these conditions there exist asymptotic solutions.

Example 2.3.7. Two zero frequencies associated with a four-dimensional Jordan block. We recall that the quadratic portion of the Hamiltonian function H for this problem is given by formula (2.73). We represent the nonquadratic terms of H as a sum of elementary monomials,

$$H = H_2 + \sum_{i_1+i_2+j_1+j_2\geq 3} h_{i_1,i_2,j_1,j_2} u_1^{i_1} u_2^{i_2} v_1^{j_1} v_2^{j_2},$$

and we consider the group of quasihomogeneous dilations satisfying (2.93):

$$u_1 \mapsto \mu^6 u_1, \quad v_1 \mapsto \mu^5 v_1, \quad u_2 \mapsto \mu^4 u_2, \quad v_2 \mapsto \mu^7 v_2.$$

From the Hamiltonian, this group selects the quasihomogeneous truncation for the problem:

$$H_{12} = \frac{\sigma}{2}(u_1^2 - 2v_1 v_2) + h_{0,3,0,0} u_2^3, \tag{2.114}$$

which is of 12th degree.

For

$$h_{0,3,0,0} \neq 0$$

the system of equations with Hamiltonian (2.114) has two asymptotic solutions

$$\mathbf{x}^{\pm}(t) = \left(\frac{u_{1_0}^{\pm}}{(\pm t)^6}, \frac{v_{1_0}^{\pm}}{(\pm t)^5}, \frac{u_{2_0}^{\pm}}{(\pm t)^4}, \frac{v_{2_0}^{\pm}}{(\pm t)^7} \right),$$

where

$$u_{1_0}^{\pm} = \frac{5600}{h_{0,3,0,0}}\sigma, \; v_{1_0}^{\pm} = \mp \frac{1120}{h_{0,3,0,0}},$$
$$u_{2_0}^{\pm} = \frac{280}{h_{0,3,0,0}}\sigma, \; v_{2_0}^{\pm} = \mp \frac{33600}{h_{0,3,0,0}}.$$

Therefore, if $h_{0,3,0,0} \neq 0$, then the system under investigation is unstable and has two asymptotic solutions, one of which converges to the equilibrium position as $t \to +\infty$, the other as $t \to -\infty$. The case of the additional degeneracy ($h_{0,3,0,0} = 0$) requires special attention. So, for the problem considered, instability is typical.

Example 2.3.8. Two zero frequencies associated with a two-dimensional Jordan block. In accordance with formula (2.72) the quadratic portion of the Hamiltonian for the problem investigated can be reduced to the form

$$H_2 = \frac{\sigma_1}{2} v_1^2 + \frac{\sigma_2}{2} v_2^2. \tag{2.115}$$

We write the nonquadratic terms of the Hamiltonian function as a sum of elementary monomials and consider a quasihomogeneous scale satisfying conditions (2.93):

$$u_j \mapsto \mu^2 u_j, \quad v_j \mapsto \mu^3 u_j, \quad j = 1, 2.$$

This scale selects the quasihomogeneous truncation of sixth degree:

$$H_6 = \frac{\sigma_1}{2} v_1^2 + \frac{\sigma_2}{2} v_2^2 + h_{3,0,0,0} u_1^3 + h_{2,1,0,0} u_1^2 u_2 + \tag{2.116}$$

$$+ h_{1,2,0,0} u_1 u_2^2 + h_{0,3,0,0} u_2^3. \tag{2.1}$$

The Hamiltonian system with the truncated Hamiltonian function (2.116) *almost always* has two asymptotic solutions

$$\mathbf{x}^{\pm}(t) = \left(\frac{u_{1_0}^{\pm}}{(\pm t)^2}, \frac{v_{1_0}^{\pm}}{(\pm t)^3}, \frac{u_{2_0}^{\pm}}{(\pm t)^2}, \frac{v_{1_0}^{\pm}}{(\pm t)^3}, \right)$$

where

$$u_{1_0}^{\pm} = u_{1_0}, \quad u_{2_0}^{\pm} = u_{2_0}, \quad v_{1_0}^{\pm} = \pm\sigma_1 u_{1_0}, \quad v_{2_0}^{\pm} = \pm\sigma_2 u_{2_0},$$

and where $\mathbf{c}_0 = (u_{1_0}, u_{2_0}) \in \mathbb{R}^2$ is a nonzero eigenvector of the quadratic vector field

$$\mathbf{f}_2 = \Bigg(-\frac{\sigma_1}{6}(3h_{3,0,0,0} u_1^2 + 2h_{2,1,0,0} u_1 u_2 + h_{1,2,0,0} u_2^2), \\ -\frac{\sigma_2}{6}(h_{2,1,0,0} u_1^2 + 2h_{1,2,0,0} u_1 u_2 + 3h_{0,3,0,0} u_2^2) \Bigg),$$

in the standard metric with eigenvalue 1, i.e.

$$\mathbf{f}_2(\mathbf{c}_0) = \mathbf{c}_0.$$

Such an eigenvector exists if, for example, the point $u_1 = u_2 = 0$ is the unique (real!) critical point of the cubic function

$$U_3(u_1, u_2) = 3h_{3,0,0,0} u_1^3 + h_{2,1,0,0} u_1^2 u_2 + h_{1,2,0,0} u_1 u_2^2 + h_{0,3,0,0} u_2^3.$$

In the four-dimensional space of coefficients of $U_3(u_1, u_2)$, the set on which the condition of uniqueness is violated has measure zero. Therefore our system is almost always unstable and has two asymptotic solutions, one of which tends to the equilibrium position as $t \to +\infty$, the other as $t \to -\infty$.

Chapter 3
Singular Problems

3.1 Asymptotic Solutions of Autonomous Systems of Differential Equations in the Critical Case of Zero Roots of the Characteristic Equation

The title of this section scarcely differs from the title of the first section of the preceding chapter. However, the present section is dedicated to the problem of the precise influence of the zero roots on the stability of a critical point and on the existence of asymptotic solutions when there exist roots of the characteristic equation of first approximation system that are *different from zero*. In order to comprehend how sharply this situation differs from that of Chap. 1, where the matrix of the linear portion of the system was nilpotent, we look at an example.

Example 3.1.1. We consider one of the simplest systems of differential equations in the plane:

$$\dot{x} = -x + y, \quad \dot{y} = -y^2. \tag{3.1}$$

The second equation in (3.1) has an obvious asymptotic solution $y(t) = t^{-1}$, that we substitute into the first equation and find the formal solution of the resulting linear nonhomogeneous equation in the form of a series:

$$x(t) = \sum_{k=1}^{\infty} x_k t^{-k}.$$

It is not difficult to compute the coefficients of this series: $x_k = (k-1)!$, i.e. the given series diverges for arbitrary finite values of t. It is likewise possible to obtain an explicit formula for the solution of this equation with "initial" condition $x(-\infty) = 0$:

$$x(t) = e^{-t} \int_{-\infty}^{t} s^{-1} e^{s}\, ds,$$

which turns out to be the Borel summation of the given divergent series [75].

It is clear that $x(t), y(t) \to 0$ as $t \to -\infty$, indicating instability.

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The reason for this phenomenon can be explained as follows. The particular solution $x(t), y(t)$ obtained lies on the center manifold of system (3.1), which is not analytic in a neighborhood of $x = y = 0$. Actually, in the case considered, the center manifold is easily computed. It is the function

$$\varphi(y) = -e^{-y^{-1}} \int_0^y e^{z^{-1}} z^{-1}\, dz = \sum_{k=1}^{\infty} (k-1)!\, y^k,$$

valid of course only for negative y.

This last series is the formal Taylor series of the function $\varphi(y)$ and can be obtained by successive integrations by parts.

Now, in Chap. 1 it was proved that the asymptotic series (1.16) are divergent in the absence of logarithms. Analogous series were constructed in Chap. 2 for systems of equations of type (2.9). In most cases these series are apt to be divergent, i.e. system (2.9) itself was obtained with the aid of a divergent normalizing transformation and is generally nonanalytic. In the example considered we have come upon the principle alternative phenomenon: the existence of a formal asymptotic solution in the form of a divergent series for an analytic system. As already stated, its appearance is connected with the nonanalytic center manifold.

We give yet another interpretation of the phenomenon considered. The problem of finding asymptotic solutions of "classical" semi-quasihomogeneous systems reduces to looking for particular solutions $\mathbf{u}(s) \to \mathbf{0}$ as $s \to 0$ of analytic systems of equations of the form

$$-\beta s \frac{d\mathbf{u}}{ds} = \mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{u}, s),$$

where $\boldsymbol{\phi}(\mathbf{u}) = O(\|\mathbf{u}\|^2)$ as $\mathbf{u} \to \mathbf{0}$ and $\boldsymbol{\psi}(\mathbf{u}, s) \to \mathbf{0}$ as $s \to 0$ (cf. (1.18) and (1.47)).

The point $s = 0$ is a regular critical point of the linear portion of this system. It is known [192] that asymptotic expansions of solutions of linear systems diverge in the neighborhood of regular critical points, and we can therefore expect that asymptotic expansions of solutions of nonlinear systems will also diverge. In system (3.1) we carry out a change in the dependent and independent variables:

$$x = t^{-1}(1+u), \quad y = t^{-1}(1+v), \quad s = t^{-1},$$

after which the system assumes the form:

$$-s^2 \frac{du}{ds} = v - u + s(1+u), \quad -s \frac{dv}{ds} = -v - v^2.$$

The right sides of this system likewise have the appearance

$$\mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{u}, s),$$

but the critical point $s = 0$ will be singular. Asymptotic expansions of solutions of linear systems in neighborhoods of singular critical points are, as a rule, divergent [192], so that in similar cases the divergence of series representing solutions of perturbed linear systems seems entirely natural. In accordance with the mentioned classification of critical points, we will attach the term *singular* to the problem of finding asymptotic solutions that are not reducible to perturbations of Fuchsian systems.

We will consider systems of differential equations with infinitely differentiable right sides, for which the origin is a critical point. We will likewise suppose that the matrices of the linearized systems are singular. We thus consider a system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \quad \dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}), \tag{3.2}$$

where $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$, $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$, $\mathbf{A}, \mathbf{B}, \mathbf{J}$ are real matrices of respective dimensions $d \times d$, $d \times n$ and $n \times n$, and the vector functions $\mathbf{f}$ and $\mathbf{g}$ are smooth mappings of the space $\mathbb{R}^d \times \mathbb{R}^n$ into $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively, whose components vanish at $(0, 0)$ along with their first order partial derivatives.

Our task amounts to finding sufficient conditions for the existence of nonexponential asymptotics to the critical point $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ under the assumption that the matrix $\mathbf{A}$ is nonsingular and that the matrix $\mathbf{J}$ is nilpotent. If the eigenvalues of the matrix $\mathbf{A}$ were to have negative real parts, we would be dealing with the so-called cases of n zero roots [133], but we will generalize the problem by assuming only that the eigenvalues of $\mathbf{A}$ are nonzero.

Without loss of generality we may assume that the matrix $\mathbf{B}$ is zero. In fact, carrying out the linear substitution $\mathbf{x} \to \mathbf{x} + \mathbf{C}\mathbf{y}$, where $\mathbf{C}$ is some unknown matrix of dimensions $d \times n$, we obtain that the linearized equations assume the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + (\mathbf{B} + \mathbf{A}\mathbf{C} - \mathbf{C}\mathbf{J})\mathbf{y}, \quad \dot{\mathbf{y}} = \mathbf{J}\mathbf{y}.$$

We consider the linear matrix equation

$$\mathbf{A}\mathbf{C} - \mathbf{C}\mathbf{J} = -\mathbf{B}. \tag{3.3}$$

As a system of $d \times n$ inhomogeneous equations in $d \times n$ unknowns, (3.3) will have a unique solution if the homogeneous equation

$$\mathbf{A}\mathbf{C} - \mathbf{C}\mathbf{J} = \mathbf{0} \tag{3.4}$$

has no nontrivial solution.

Since $\det \mathbf{A} \neq \mathbf{0}$ and the operator $\mathbf{J}$ is nilpotent, the matrices $\mathbf{A}$ and $\mathbf{J}$ will not have common eigenvalues, so that (3.4) has only the trivial solution [66].

Thus we proceed further with the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \quad \dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}). \tag{3.5}$$

Since our problem is that of finding the precise effect of *zero* roots on the existence of asymptotic solutions, it would be natural to try to select from Eq. (3.5) some "critical" subsystem, for which the matrix of the linear approximation would be nilpotent. The possibility of such an extended system of (3.5), at least on a formal level, is guaranteed by the following theorem.

Theorem 3.1.1. *If in system (3.5) the matrix* $\mathbf{A}$ *is nonsingular and* $\mathbf{J}$ *is nilpotent, then (3.5) has a formal invariant manifold*

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{y}), \tag{3.6}$$

where φ *is some vector function whose expansion into a formal Maclaurin series in the neighborhood of* $y = 0$ *begins with quadratic or higher order terms.*

Proof. We consider the nonsingular change of variable

$$\mathbf{x} = \mathbf{z} + \boldsymbol{\varphi}(\mathbf{y}),$$

subsequent to which the first group of equations in (3.5) assumes the form

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \widetilde{\mathbf{f}}(\mathbf{z}, \mathbf{y}),$$

where

$$\widetilde{\mathbf{f}}(\mathbf{z}, \mathbf{y}) = \mathbf{A}\boldsymbol{\varphi}(\mathbf{y}) + \mathbf{f}(\mathbf{z} + \boldsymbol{\varphi}(\mathbf{y}), \mathbf{y}) - \mathbf{d}\boldsymbol{\varphi}(\mathbf{y})\,(\mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{z} + \boldsymbol{\varphi}(\mathbf{y}), \mathbf{y}))\,.$$

We try to choose $\boldsymbol{\varphi}(\mathbf{y})$ so that the plane $\mathbf{z} = \mathbf{0}$ will be an invariant manifold, i.e. $\widetilde{\mathbf{f}}(\mathbf{0}, \mathbf{y}) \equiv \mathbf{0}$. From this we get a system of partial differential equations for determining $\boldsymbol{\varphi}(\mathbf{y})$:

$$\mathbf{A}\boldsymbol{\varphi}(\mathbf{y}) + \mathbf{f}(\boldsymbol{\varphi}(\mathbf{y}), \mathbf{y}) - \mathbf{d}\boldsymbol{\varphi}(\mathbf{y})\,(\mathbf{J}\mathbf{y} + \mathbf{g}(\boldsymbol{\varphi}(\mathbf{y}), \mathbf{y})\,) = \mathbf{0}. \tag{3.7}$$

We will seek $\boldsymbol{\varphi}(\mathbf{y})$ in the form of a Maclaurin series

$$\boldsymbol{\varphi}(\mathbf{y}) = \sum_{m=2}^{\infty} \boldsymbol{\varphi}_m(\mathbf{y})\,.$$

The corresponding forms in the given expansion can be subsequently found by induction. From (3.7) it follows that each homogeneous form $\boldsymbol{\varphi}_m(\mathbf{y})$ satisfies a linear system of equations in the partial derivatives,

$$\mathbf{A}\boldsymbol{\varphi}_{\mathbf{m}}(\mathbf{y}) + d\,\boldsymbol{\varphi}_{\mathbf{m}}(\mathbf{y})\mathbf{J}\mathbf{y} = \boldsymbol{\vartheta}_{\mathbf{m}}(\mathbf{y}),$$

where the $\boldsymbol{\vartheta}_m$ are homogeneous vector functions of degree m, dependent on the "preceding" forms $\boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_{m-1}$.

In this way, if (3.8) is solved for the arbitrary forms $\boldsymbol{\vartheta}_m$, then all the forms $\boldsymbol{\varphi}_m$ could be computed by induction. Further, let the matrix $\mathbf{A}$ be reduced to Jordan form. It is then easy to see that it is sufficient to establish the solvability of (3.8) for the case where $\mathbf{A}$ is a Jordan block, on whose diagonal $\lambda \neq 0$ appears. We rewrite (3.8) in terms of coordinates. Let $\boldsymbol{\varphi}_m = (\varphi_m^1, \ldots, \varphi_m^n)$, $\boldsymbol{\vartheta} = (\vartheta_m^1, \ldots, \vartheta_m^n)$. Then

$$\lambda\varphi_m^j(\mathbf{y}) + \boldsymbol{\varphi}_{\mathbf{m}}^{\mathbf{j+1}}(\mathbf{y}) - \langle d\boldsymbol{\varphi}_{\mathbf{m}}^{\mathbf{j}}(\mathbf{y}), \mathbf{Jy}\rangle = \boldsymbol{\vartheta}_{\mathbf{m}}^{\mathbf{j}}(\mathbf{y}), \quad j = 1, \ldots, n-1$$

$$\lambda\varphi_m^n(\mathbf{y}) - \langle d\boldsymbol{\varphi}_{\mathbf{m}}^{\mathbf{n}}(\mathbf{y}), \mathbf{Jy}\rangle = \boldsymbol{\vartheta}_{\mathbf{m}}^{\mathbf{n}}(\mathbf{y}).$$

It is clearly sufficient to establish the solvability of the last equation: the remaining $n-1$ equations are solved by induction in an analogous fashion. The space $\mathfrak{J}_m$ of scalar homogeneous forms on $\mathbb{R}^n$, whose degree doesn't exceed m, is finite dimensional. Therefore, for the solvability of the corresponding equation, it is sufficient that Ker $D_\lambda = \{0\}$, where $D_\lambda = \lambda + D$, $D = \langle d, \mathbf{Jy}\rangle$.

We prove that the operator D is nilpotent. Then $-\lambda$ can't be its eigenvalue and the kernel of the operator D_λ is zero. Let the matrix $\mathbf{J}$ be reduced to Jordan form. It is then sufficient to prove the nilpotence of the operator D for the case where $\mathbf{J}$ is a Jordan block with zero diagonal. In this case

$$D = y^2\frac{\partial}{\partial y^1} + \ldots + y^n\frac{\partial}{\partial y^{n-1}}.$$

The elementary monomials

$$\left\{(y^1)^{i_1}\ldots(y^n)^{i_n}, i_1 + \ldots + i_n \le m\right\}$$

form a basis for the space of homogeneous forms of degree not exceeding m. The action of the operator D on this basis is determined by the formula

$$D\left((y^1)^{i_1}\ldots(y^n)^{i_n}\right) = \sum_{p=1}^{m-1} i_p(y^1)^{i_1}\cdots(y^p)^{i_p-1}(y^{p+1})^{i_{p+1}+1}\cdots(y^n)^{i_n}.$$

The subspace spanned by the monomials $(y^n)^{i_n}$ forms the kernel of the operator D. We consider the subspace spanned by the basic monomials of form $(y^{n-1})^{i_{n-1}}(y^n)^{i_n}$. It is clear that the $(i_{n-1}+1)$–th application of the operator D to these basic monomials leads to their vanishing.

We proceed further by induction. We now consider the invariant subspace spanned by basic monomials containing only the last $n-p+1$ variables. These monomials are represented in the form $(y^p)^{i_p}\varphi^{(p)}(y^{p+1}, \ldots, y^n)$. Suppose that all $\varphi^{(p)} \in$ Ker D^{S_p} for some sufficiently large $S_p \in \mathbb{N}$. After the S_p–th application of the operator D we obtain a polynomial whose degree with respect to y^p is reduced to unity. Therefore the $(i_p+1)S_p$–th application of the operator D leads to its vanishing and so the subspace considered lies in the kernel of some S_{p+1}–th power of D. Using the given procedure, the entire subspace $\mathfrak{J}_m$ can be exhausted, which proves the nilpotence of D.

Consequently, at each m–th step of the induction the above system of partial differential equations is solved.

The theorem is proved.

If none of the eigenvalues of the matrix $\mathbf{A}$ lies on the imaginary axis, then this result is a consequence of the center manifold theorem [137]. But this result might be obtained from the Poincaré-Dulac theorem on the normal form [7], making use of the fact that the nonsingularity of the matrix $\mathbf{A}$ in essence indicates an absence of resonances of a specific type. Such an approach was used, for example, in the paper [25] for constructing a formal invariant manifold of type (3.6) of a certain Hamiltonian system of a special form. However, reduction to normal form requires the transformation of all the phase variables, which makes the corresponding algorithm for the construction of the invariant manifold all the more complicated. The method we have considered here allows us likewise to obtain *explicit* equations for the vector function $\boldsymbol{\varphi}(\mathbf{y})$ that yields the desired manifold.

The formal equations of motion of the system on the constructed invariant manifold will assume the form

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{g}(\boldsymbol{\varphi}(\mathbf{y}), \mathbf{y}) = \mathbf{h}(\mathbf{y}). \tag{3.8}$$

All the eigenvalues of the matrix for the linear approximating system (3.8) are zero, whence the application to it of the formal method for constructing asymptotic solutions was worked out in Chap. 1. The following result holds, generalizing a theorem from [59].

Theorem 3.1.2. *Let the reduction of system (3.8) on the formal invariant manifold (3.6) be semi-quasihomogeneous with respect to the structure induced by some diagonal matrix G whose diagonal elements are positive. Further let the quasi-homogeneous truncation*

$$\dot{\mathbf{y}} = \mathbf{h}_{\mathrm{q}}(\mathbf{y}) \tag{3.9}$$

of system (3.8) be such that there exists a nonzero vector $y_0^\gamma \in \mathbb{R}^n$ and a number $\gamma = \pm 1$, satisfying the algebraic system of equations

$$-\gamma \mathbf{G}\mathbf{y}_0^\gamma = \mathbf{h}_{\mathrm{q}}(\mathbf{y}_0^\gamma). \tag{3.10}$$

Then the system of differential equations (3.2) has an asymptotic solution $(x(t), y(t)) \to (0,0)$ as $t \to \gamma \times \infty$.

As a consequence of Theorem 3.1.2, we have a theorem on the instability of the critical point of the system (3.2):

Theorem 3.1.3. *Let all the conditions of the preceding theorem be fulfilled and let $\gamma = -1$ in the system of differential equations (3.10). Then the critical point $x = 0$, $y = 0$ of system (3.2) is unstable.*

Proof. It follows from Theorem 1.1.2 that system (3.8) has a particular asymptotic solution of the form

$$\mathbf{y(t)} = \sum_{\mathbf{k=0}}^{\infty} \mathbf{y_k} \left(\ln(\boldsymbol{\gamma}\mathbf{t})\right) (\boldsymbol{\gamma}\mathbf{t})^{-\mathbf{b_k}}, \tag{3.11}$$

where $b_k = g_k + \beta k > 0$ is an increasing sequence of real numbers, where the g_k are elements of the matrix $\mathbf{G}$, and where the $\mathbf{y_k}$, as usual, are some polynomial vector functions.

Putting this solution in the equation of the formal invariant manifold (3.6), we get a formal series for $\mathbf{x}$:

$$\mathbf{x(t)} = \boldsymbol{\varphi}\left(\mathbf{y(t)}\right) = \sum_{\mathbf{k=0}}^{\infty} \mathbf{x_k} \left(\ln(\boldsymbol{\gamma}\mathbf{t})\right) (\boldsymbol{\gamma}\mathbf{t})^{-\mathbf{a_k}}. \tag{3.12}$$

Here a_k is likewise an increasing sequence of real numbers and the $\mathbf{x_k}$ are polynomial vector functions.

If we could prove the convergence of the series (3.11) and (3.12), then the theorem would be proved, since each term of these series tends to zero as $t \to \gamma\times\infty$. However, as Example 3.1.1 showed, these series can diverge (and, as a rule, *do* in fact diverge!). The technique—based on the apparatus of the implicit function theorem—that was used in the first chapter for the proof of existence of *smooth* solutions, is also inapplicable in this situation. However, Kuznetsov's theorem [125, 126], which will be examined more closely in Sect. 3.3 of this chapter, asserts that the existence of formal solutions is implied by the existence of infinitely differentiable solutions with corresponding asymptotic. At this stage we will be limited to just confirming the fact of the existence of formal solutions.

The theorem is proved.

In the theorem formulated in [59] that is analogous to Theorem 3.1.2, there is an additional condition on the absence of resonances of a specified type between the eigenvalues of the matrix $\mathbf{A}$, up to some sufficiently high order. This additional condition arises because an asymptotic solution is sought not just on the invariant manifold, but in all of phase space. This procedure requires the imposition of additional conditions for determining a critical subsystem (of finite order) in normal form and, consequently, an additional nonresonance condition.

The idea of constructing invariant subspaces for investigating stability in the critical case of zero roots goes back to Lyapunov [133], who used exactly such methods. Here, however, we are not imposing any requirements, so that all remaining eigenvalues of the first order approximation matrix have negative real part. Consider, for example, a situation that was already discussed in the second chapter, where some of the roots of the characteristic equation of the first approximation system are zero and the remainder are pure imaginary. The general algorithm for investigating stability in the "pure imaginary" situation calls for reduction to Poincaré normal form and "quasihomogeneous" analysis of a system of dimension $d + n$. It is clear, from the computational point of view, that the algorithm for constructing the formal invariant manifold and the subsequent analysis of the $(d + n)$–dimensional system are substantially economical. Moreover, we need just recall that instability in a system can be caused by interactions of modes corresponding to the zero roots with modes corresponding to the pure imaginary roots. It is reasonable that in this case the procedure for reducing the system on the formal invariant manifold (3.6) doesn't yield sufficient conditions for instability.

Example 3.1.2. We show how to obtain simple criteria for the instability of an equilibrium position that is stable for the first approximation of the multidimensional Hamiltonian system in the case of one null frequency. This case was studied by A.G. Sokol'skiy [178] in the absence of resonance between the remaining frequencies. The last requirement originates from Sokol'skiy's method, which provides reduction of the system to Birkhoff normal form. In order that, in the multidimensional case in the presence of some resonance, the normal form have a concrete aspect suitable for study, it is essential to have conditions for excluding additional resonances. The technique of reduction on the formal invariant manifold allows us to avoid additional complexity in the reduction to normal form.

We thus consider a Hamiltonian system with n degrees of freedom and with Hamiltonian function

$$\begin{aligned} H = & \tfrac{1}{2}\sum\nolimits_{j=1}^{n-1} \sigma_j \omega_j (u_j^2 + v_j^2) + \\ & + H_M(u_1, \ldots, u_{n-1}, u_n, v_1, \ldots, v_{n-1}, v_n) + \ldots, \end{aligned} \tag{3.13}$$

where $\sigma_j = \pm 1$, $\omega_j > 0$, $j = 1, \ldots, n-1$, H_M $(M \geq 3)$ is the first nontrivial form in the expansion of the Hamiltonian function in a Maclaurin series after the quadratic, and where the dots denote the totality of nonlinear terms of order higher than M.

We consider the function of two variables

$$h_M(u_n, v_n) = H_M(0, \ldots, 0, u_n, 0, \ldots, 0, v_n).$$

We have:

Theorem 3.1.4. *If the function $h_M(u_n, v_n)$ is skew-symmetric and its set of zeros does not coincide with the set of zeros of its gradient, then the equilibrium position $\mathbf{u} = \mathbf{v} = \mathbf{0}$ of the system with Hamiltonian function (3.13) is unstable.*

Proof. The equations of motion of the system with Hamiltonian (3.13) have the form

$$\begin{aligned} &\dot{u}_j = -\sigma_j \omega_j v_j - \frac{\partial}{\partial v_j} H_M + \ldots, \quad \dot{v}_j = \sigma_j \omega_j u_j + \frac{\partial}{\partial u_j} H_M + \ldots, \\ &\qquad j = 1, \ldots, n-1 \\ &\dot{u}_n = -\frac{\partial}{\partial v_n} H_M + \ldots, \qquad\qquad \dot{v}_n = \frac{\partial}{\partial u_n} H_M + \ldots \end{aligned} \tag{3.14}$$

This system of equations has the form of (3.5), so that Theorem 3.1.1 applies to the system. The equation of the formal invariant manifold (3.6), in the concrete case at hand, has the form

$$u_j = \varphi_j(u_n, v_n), \quad v_j = \psi_j(u_n, v_n), \quad j = 1, \ldots, n-1,$$

where the functions φ_j, ψ_j, along with their first derivatives, vanish at $u_n = 0$, $v_n = 0$.

The system, reduced on the formal invariant manifold, then takes the form

$$\dot{u}_n = -\frac{\partial}{\partial v_n} h_M + \ldots, \quad \dot{v}_n = -\frac{\partial}{\partial u_n} h_M + \ldots,$$

where the dots denote the totality of terms of order greater than $M-1$.

We select a *homogeneous* truncation for this system in a natural way. By keeping only order $M-1$ terms, we obtain a Hamiltonian system with a single degree of freedom and Hamiltonian h_M. In fact, we studied this system in Sect. 2.3 of Chap. 2. If the conditions of the theorem are satisfied, then this system has both "entering" and "exiting" particular solutions in the form of twisted rays. Therefore, by Theorem 3.1.2, the full system (3.14) has particular solutions that tend to the equilibrium position both as $t \to +\infty$ and as $t \to -\infty$, which indicates instability.

The theorem is proved.

The next problem we consider occupies an intermediate position between regular and singular problems.

We now let all eigenvalues of the matrix $\mathbf{A}$ have negative real parts and study an asymptotic solution of (3.2) that tends to the critical point as $t \to +\infty$. System (3.2) has a d–parameter family of particular solutions having exponential asymptotic. These solutions may be expanded in Lyapunov series in which will appear exponentials of the time t with negative real degrees and whose coefficients will depend polynomially on t, along with trigonometric functions of a certain set of quantities $\omega_p t,\ p \leq d$. On the other hand, with satisfaction of the hypothesis of Theorem 3.1.2, the system (3.2) will have solutions with series expansions of the form (3.12), containing t raised to negative real powers, whose coefficients depend polynomially on $\ln t$. Both types of solutions lie on the invariant manifold of the system (stable and center, respectively). A natural question then arises: "how do these two types of motion interact with each other?" We show that with some additional assumptions there exist particular solutions of the system which lie neither on the stable nor on the center manifold, whose expansions contain e^t and t to negative powers. An elementary example of such an expansion can be seen in Sect. 1.1 of Chap. 1 (Example 1.5.5).

Here is a less trivial example.

Example 3.1.3. The system of differential equations

$$\dot{x} = -x(x+1)(y+1), \quad \dot{y} = -y^2 \tag{3.15}$$

has two obvious invariant manifolds: a stable ($y = 0$) and a center ($x = 0$).

All the asymptotic solutions lying on the stable manifold of system (3.15) expand into exponential series

$$x(t) = \frac{ce^{-t}}{1-ce^{-t}} = \sum_{k=1}^{\infty} c^k e^{-kt}, \quad y(t) = 0.$$

There likewise exists a unique asymptotic solution lying on the center manifold:

$$x(t) = 0, \quad y(t) = t^{-1}.$$

It is not difficult, however, to see that system (3.15) has a one-parameter family of solutions that do not belong to the given manifold and expand into series which are a special hybrid of the series of Lyapunov and Laurent:

$$x(t) = \frac{ce^{-t}t^{-1}}{1-ce^{-t}t^{-1}} = \sum_{k=1}^{\infty} c^k e^{-kt} t^{-k}, \quad y(t) = t^{-1}.$$

We return to the analysis of system (3.5) and assume that the following conditions are fulfilled:

1. $\Re\lambda_j < 0$, $j = 1, \ldots, d$, where the λ_j are the roots of the characteristic equation $\det(\mathbf{A} - \boldsymbol{\lambda}\mathbf{E}) = \mathbf{0}$,
2. The matrix $\mathbf{A}$ is diagonalizable,
3. The expansion of the right side of the second part of (3.7) begins with terms that are at least of second order, namely $d_{\mathbf{y}}\mathbf{g}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

We consider the system (3.5) and suppose that it has been reduced to Poincaré normal form [7, 41]. For the normalized variables, we keep the old notations $(\mathbf{x}, \mathbf{y})$. Since all the eigenvalues of the matrix $\mathbf{A}$ have negative real part, there are no resonances of the type

$$\sum_{j=1}^{d} p_j \lambda_j = 0, \quad p_j \in \mathbb{N} \cup \{0\}, \quad j = 1, \ldots, d, \quad \sum_{j=1}^{d} p_j \geq 2.$$

Then the vector function $\mathbf{g}$ doesn't depend on $\mathbf{x}$. As for the vector function $\mathbf{f}$, by definition of a normal form we have the identity

$$\mathbf{f}(\exp(\mathbf{A}\boldsymbol{\sigma})\mathbf{x}, \mathbf{y}) = \exp(\mathbf{A}, \boldsymbol{\sigma})\mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{3.16}$$

for any $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{\mathbf{d}+\mathbf{n}}$, $\sigma \in \mathbb{R}$, and furthermore:

$$\mathbf{f}(\mathbf{0}, \mathbf{y}) \equiv \mathbf{0}.$$

So we are dealing with a system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \quad \dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}), \tag{3.17}$$

such that the vector function $\mathbf{f}(\mathbf{x}, \mathbf{y})$ satisfies (3.16).

For system (3.17), the d–dimensional subspace $\mathbf{y} = \mathbf{0}$ is the stable manifold, and the n–dimensional subspace $\mathbf{x} = \mathbf{0}$ is center manifold. We impose yet one additional condition:

$$d_{\mathbf{y}}\mathbf{f}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{0}, \tag{3.18}$$

i.e. the right sides of the first subsystem in (3.17) contain only those components of the vector $\mathbf{y}$ that are of degree two or greater.

We consider the expansions

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{m}=1} \mathbf{f}_{\mathbf{m}+1}(\mathbf{x}, \mathbf{y}), \quad \mathbf{g}(\mathbf{y}) = \sum_{\mathbf{m}=1} \mathbf{g}_{\mathbf{m}+1}(\mathbf{y}),$$

where $\mathbf{f}_{m+1}, \mathbf{g}_{m+1}$ are homogeneous vector functions of $\mathbf{y}$ of degree $m + 1$, in which the coefficients of the form $\mathbf{f}_{m+1}$ are formal series in $\mathbf{x}$.

We will further suppose that

$$\mathbf{g}_2(\mathbf{y}) \not\equiv \mathbf{0}.$$

In the $(d + n + 1)$–dimensional extended phase space of the system, we consider the system of differential equations

$$\frac{d\mathbf{x}}{d\sigma} = t\mathbf{A}\mathbf{x}, \quad \frac{d\mathbf{y}}{d\sigma} = -\mathbf{y}, \quad \frac{dt}{d\sigma} = t. \tag{3.19}$$

System (3.19) generates the flow

$$\mathbf{x} = \exp\left(\mathbf{A}\tau(e^{\sigma} - 1)\right)\boldsymbol{\xi}, \quad \mathbf{y} = e^{-\sigma}\boldsymbol{\eta}, \quad t = e^{\sigma}\tau. \tag{3.20}$$

Using (3.20) in the variables $\mathbf{x}, \mathbf{y}, \tau$, we pass to new variables $\boldsymbol{\xi}, \boldsymbol{\eta}, \tau$. From (3.16) it follows that the system (3.17) considered assumes the form

$$\begin{gathered} \boldsymbol{\xi}' = \mathbf{A}\boldsymbol{\xi} + \sum_{\mathbf{m}=1} e^{-m\sigma}\mathbf{f}_{\mathbf{m}+1}(\boldsymbol{\xi}, \boldsymbol{\eta}), \quad \boldsymbol{\eta}' = \sum_{\mathbf{m}=0} e^{-m\sigma}\mathbf{g}_{\mathbf{m}+2}(\boldsymbol{\eta}), \\ (\cdot)' = d/dt. \end{gathered} \tag{3.21}$$

From (3.21) it is clear that the flow (3.20) generated by the system of equations (3.19) is exponentially asymptotic under the generalized symmetry group of system (3.17). The truncated system has here the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \dot{\mathbf{y}} = \mathbf{g}_2(\mathbf{y}). \tag{3.22}$$

In a rather general situation the system (3.22) has a one-parameter family of particular solutions that lie on an orbit of the group (3.20) and have the form

$$\mathbf{x} = \exp\left(\mathbf{A}(t-\tau_0)\right)\mathbf{c}, \quad \mathbf{y} = \mathbf{y_0}(t/\tau_0)^{-\mathbf{1}}, \quad \tau_0 > 0.$$

Here $\mathbf{c} \in \mathbb{R}^{\mathbf{d}}$ is a vector of parameters for the family and $\mathbf{y_0} \in \mathbb{R}^{\mathbf{n}}$ must satisfy the system of equations

$$\tau_0 \mathbf{g}_2(\mathbf{y_0}) = -\mathbf{y_0}. \tag{3.23}$$

We can thus apply Theorem 1.5.2. We introduce a new "time" σ to the system:

$$\frac{d\xi}{d\sigma} = \tau_0 \sum_{m=1} e^{-m\sigma}\mathbf{f}_{m+1}(\boldsymbol{\xi}, \boldsymbol{\eta}), \quad \frac{d\eta}{d\sigma} = \boldsymbol{\eta} + \tau_0 \sum_{m=0} e^{-m\sigma}\mathbf{g}_{m+2}(\boldsymbol{\eta}). \tag{3.24}$$

Without loss of generality we can set the parameter τ_0 equal to $+1$.

The system (3.24) has a family of formal particular solutions

$$\boldsymbol{\xi}(\sigma) = \sum_{k=0}^{\infty} \boldsymbol{\xi}_k(\sigma)e^{-k\sigma}, \quad \boldsymbol{\eta}(\sigma) = \sum_{k=0}^{\infty} \boldsymbol{\eta}_k(\sigma)e^{-k\sigma},$$

where $\boldsymbol{\eta}_0 = \mathbf{y_0}$ and the vector $\boldsymbol{\xi}_0$ can be chosen arbitrarily.

Here $\boldsymbol{\xi}_k(\sigma)$, $\boldsymbol{\eta}_k(\sigma)$ is a polynomial vector function of σ. Returning to the original variables, we find that the system (3.17) has a family of formal particular solutions of the form

$$\mathbf{x}(\mathbf{t}) = \exp(\mathbf{A}\mathbf{t}) \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \mathbf{x_k}(\ln \mathbf{t})\mathbf{t}^{-\mathbf{k}}, \quad \mathbf{y}(\mathbf{t}) = \mathbf{t}^{-\mathbf{1}} \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \mathbf{y_k}(\ln \mathbf{t})\mathbf{t}^{-\mathbf{k}}, \tag{3.25}$$

where the coefficients $\mathbf{x_k}, \mathbf{y_k}$ are polynomials in $\ln t$.

If, from the normal coordinates, we return to the original variables (retaining the notation $(\mathbf{x}, \mathbf{y})$ for them), then the solutions of (3.2) will no longer have such a simple form as in (3.25), since the exponents will be "shuffled". We consider a simple case where all eigenvalues of the matrix $\mathbf{A}$ are real. Then in the original variables the system (3.2) will have the formal solution

$$\begin{pmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \mathbf{x}_k(t) \\ \mathbf{y}_k(t) \end{pmatrix} e^{-a_k t}, \tag{3.26}$$

where the coefficients $\mathbf{x}_k(t), \mathbf{y}_k(t)$ can be represented by the series

$$\begin{pmatrix} \mathbf{x}_k(t) \\ \mathbf{y}_k(t) \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} \mathbf{x}_{k_j}(t) \\ \mathbf{y}_{k_j}(t) \end{pmatrix} t^{-j},$$

and the sequence of exponents $\{a_k\}_{k=0}^{\infty}$, $a_0 \geq 0$ tends monotonically to $+\infty$.

In the last series the coefficients $\mathbf{x}_{k_j}, \mathbf{y}_{k_j}$ are polynomially dependent on the time t.

We thus have:

Theorem 3.1.5. *Suppose that in the system (3.2) the matrix* **A** *is diagonalizable, that all its eigenvalues are negative real numbers and that the matrix* **J** *is zero. Let the normal form (3.17) of system (3.5) be such that (3.18) holds and that the quadratic part of the vector field* **g**, *when the latter is expanded into a Maclaurin series, has a nonzero eigenvector (see (3.23)). Then there exists a d–parameter family of smooth and asymptotic (as $t \to +\infty$) solutions of system (3.2), represented in the form of the series (3.26).*

At this stage we have proved only the existence of formal solutions. The constructed series could diverge, for the reason at least that the normalizing transformation used in its construction is, as a rule, divergent. On the other hand, as is shown by Example 3.1.3, in a number of cases these series can converge for sufficiently large t. The proof of the fact that these formal solutions correspond to actual smooth solutions is left for Sect. 3.3.

3.2 Concerning Iterated Logarithms

In order to better understand the discussion, we first consider some simple but instructive examples.

Example 3.2.1. The system of differential equations in the plane,

$$\dot{x} = -x^2, \quad \dot{y} = -xy^2, \tag{3.27}$$

can be treated in two ways. First, it is semi-homogeneous (i.e. semi-quasihomogeneous with respect to the structure imposed by the matrix $\mathbf{G} = \mathbf{E}$). Its truncation has the very simple form

$$\dot{x} = -x^2, \quad \dot{y} = 0.$$

An asymptotic particular solution (as $t \to +\infty$) of system (3.27), represented in the form of series (1.16) within a time shift, has the form

$$x(t) = \frac{1}{t}, \quad y(t) = 0. \tag{3.28}$$

On the other hand, this system is quasihomogeneous with respect to the structure whose matrix is $\mathbf{G} = \operatorname{diag}(\mathbf{1}, \mathbf{0})$. This structure allows us to find the solution (3.28) without our having to resort to the series expansion (1.16), since it lies on an orbit of the corresponding group of dilations.

Besides the solution (3.28), system (3.27) has yet another asymptotic solution,

$$x(t) = \frac{1}{t}, \quad y(t) = \frac{1}{\ln t}, \tag{3.29}$$

whose nature isn't yet clear.

We make a change of variables, perturbing solution (3.28):

$$x = t^{-1}(1 + u), \quad y = v, \quad \tau = \ln t.$$

By semi-quasihomogeneity, (3.27) is transformed by this change to the autonomous system

$$u'^2, \quad v'^2 - v^2 u, \quad (\cdot)' = \frac{d}{d\tau}(\cdot). \tag{3.30}$$

The linear approximation matrix of this system (being the Kovalevsky matrix for the solution (3.28)) has eigenvalues $\rho_1 = -1$, $\rho_2 = 0$. We now apply the theorem on the center manifold to system (3.30). This manifold in the present case has the very simple form

$$u = 0,$$

so that system (3.30) is written in the form

$$v' = -v^2.$$

and has the obvious asymptotic solution $v(\tau) = \tau^{-1}$, corresponding to the second component of (3.29).

It is not difficult to present an example of a system, based on system (3.27), that has an asymptotic solution containing reciprocal powers of the iterated logarithm of the time. For example, the system

$$\dot{x} = -x^2, \quad \dot{y} = -xy^2, \quad \dot{z} = -xyz^2,$$

which is quasihomogeneous with respect to the structure from the matrix

$$\mathbf{G} = \operatorname{diag}(\mathbf{1}, \mathbf{0}, \mathbf{0}),$$

has the particular asymptotic solution (as $t \to +\infty$)

$$x(t) = \frac{1}{t}, \quad y(t) = \frac{1}{\ln t}, \quad z(t) = \frac{1}{\ln(\ln t)}.$$

Example 3.2.2. We "spoil" system (3.27) by making it *semi-quasihomogeneous* with respect to the structure from the matrix $\mathbf{G} = \operatorname{diag}(\mathbf{1}, \mathbf{0})$:

$$\dot{x} = -x^2, \quad \dot{y} = -xy^2 - x^2y^2. \tag{3.31}$$

This system is also semi-quasihomogeneous and admits the solution

$$x(t) = \frac{1}{t}, \quad y(t) = 0.$$

System (3.31) also has a less obvious asymptotic solution

$$x(t) = \frac{1}{t}, \quad y(t) = \frac{1}{\ln t - t^{-1}} = \frac{1}{\ln t}\sum_{k=0}^{\infty}(t \ln t)^{-k}. \tag{3.32}$$

If in Eq. (3.31) we make the substitution $x = t^{-1}(1 + u)$, $y = v$, $\tau = \ln t$, then the system is rewritten in the form

$$u' = -u - u^2, \quad v' = -v^2(1 + u)(1 + e^{-\tau}(1 + u)). \tag{3.33}$$

Although the theorem on the center manifold isn't applicable to system (3.33), the line $u = 0$ is invariant as before. The system, reduced on this line, has the appearance

$$v' = -(1 + e^{-\tau})v^2.$$

This last system has separated variables and is thus easily integrated. We note, however, that the change of dependent variable

$$v = \frac{w}{1 - e^{-\tau}w}$$

reduces the given system to autonomous form:

$$w' = -w^2.$$

This system has a particular asymptotic solution, i.e. the corresponding y–component of the solution (3.32). The transformation thus constructed (dependent on logarithmic time τ) is in a certain sense a normalizing transformation, "killing off" the time dependency of the right sides of the system (3.33) on the invariant manifold $u = 0$.

From the examples considered we can draw a number of conclusions. First, that reciprocal powers of logarithms (both ordinary and iterated) appear in the expansions of asymptotic solutions when there are zeros among the eigenvalues of the Kovalevsky matrix, a situation that may be encountered if the critical point of the truncated system isn't isolated. Second, the kernel of the linear operator that corresponds to the Kovalevsky matrix is tangent to some invariant manifold of the "system of equations of perturbed motion", on which the asymptotic solution lies. Finally, the system reduced on this manifold can, evidently, be reduced to autonomous form.

The following example shows that, because of weaker than exponential convergence of some components of an asymptotic solution to a singular critical point, eigenvalues of the Kovalevsky matrix may be not only zero, but also pure imaginary.

Example 3.2.3. We consider the system of equations

$$\dot{x} = -x^2, \quad \dot{y} = -x\left(z + \tfrac{1}{2}(y^2 + z^2)y\right),$$
$$\dot{z} = -x\left(-y + \tfrac{1}{2}(y^2 + z^2)z\right),$$

which is quasihomogeneous with respect to the matrix $\mathbf{G}$ =diag $(1, 0, 0)$.

This system has the following particular solution:

$$x(t) = \frac{1}{t}, \quad y(t) = \frac{\cos \ln t}{\sqrt{\ln t}}, \quad z(t) = \frac{\sin \ln t}{\sqrt{\ln t}}.$$

It is easy to compute the eigenvalues of the Kovalevsky matrix that correspond to the solution $x(t) = t^{-1}$, $y(t) = z(t) = 0$ lying on an orbit of the corresponding group of quasihomogeneous dilations: $\rho_1 = -1$, $\rho_{2,3} = \pm i$.

Here we consider in detail only the question of the influence of zero eigenvalues of the Kovalevsky matrix on the behavior of asymptotic solutions in the neighborhood of a strongly singular critical point.

We thus consider a smooth system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{3.34}$$

for which the origin $\mathbf{x} = \mathbf{0}$ is a singular critical point (the Jacobian matrix $d\mathbf{f}(\mathbf{0})$ has only zero eigenvalues) that is positive semi-quasihomogeneous with respect to the structure induced by the matrix $\mathbf{G}$, whose eigenvalues have nonnegative real part.

Let the truncated system

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x})$$

have the particular asymptotic solution

$$\mathbf{x}^\gamma(t) = (\gamma t)^{-\mathbf{G}} \mathbf{x}_\mathbf{0}^\gamma.$$

It is clear that, after the substitution

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_\mathbf{0}^\gamma + \mathbf{u}(\gamma t)), \quad \tau = \ln(\gamma t),$$

system (3.34) is rewritten in the following form:

$$\mathbf{u}' = \mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{u}, \tau), \tag{3.35}$$

where $\boldsymbol{\phi}(\mathbf{u}) = O(\|\mathbf{u}\|^2)$ as $\mathbf{u} \to \mathbf{0}$, $\boldsymbol{\psi}(\mathbf{u})$ represented in the form of a power series in $e^{-\beta\tau}$ without free term, the dash denoting differentiation with respect to logarithmic time τ.

System (3.35) is asymptotically autonomous, i.e. time dependence disappears as $\tau \to +\infty$.

We show that there exists a *formal*, asymptotically identical transformation $\mathbf{u} \mapsto \mathbf{z}$, reducing system (3.35) to autonomous form. More precisely, we have:

Theorem 3.2.1. *Using a formal transformation of the form*

$$\mathbf{u} = \hat{\mathbf{u}}(\tau) + \mathbf{B}(\tau)\mathbf{z} + \sum_{\mathbf{m}=\mathbf{2}}^{\infty} \mathbf{U}_{\mathbf{m}}(\mathbf{z},\tau), \tag{3.36}$$

where the vector function $\hat{\mathbf{u}}(\tau)$ is some formal series

$$\hat{\mathbf{u}}(\tau) = \sum_{k=1}^{\infty} \mathbf{u}_k(\tau) e^{-k\beta\tau}, \tag{3.37}$$

where the $\mathbf{u}_k(\tau)$ are vector polynomials of τ, where the matrix $\mathbf{B}(\tau)$ is represented as a series

$$\mathbf{B}(\boldsymbol{\tau}) = \mathbf{E} + \sum_{k=1}^{\infty} \mathbf{B}_k(\tau) e^{-k\beta\tau}, \tag{3.38}$$

where $\mathbf{B_k}(\tau)$ is a matrix polynomial in τ, and where the form $\mathbf{U}_m(z,\tau)$ is homogeneous in z of degree $m = 2, 3, \ldots$, whose coefficients are formal power series in $e^{-\beta\tau}$ without free terms, whose coefficients in turn are polynomially dependent on the logarithmic time τ, then the system of equations (3.35) can be reduced to the autonomous form

$$\mathbf{z}' = \mathbf{K}\mathbf{z} + \boldsymbol{\phi}(\mathbf{z}). \tag{3.39}$$

Proof We build the transformation (3.36) in several steps.

First step. We send the critical point to the coordinate origin. We have:

Lemma 3.2.1. *With the aid of the shift $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{v}$, where $\hat{\mathbf{u}}(\tau)$ is a formal series of form (3.37), system (3.35) can be rewritten in the form*

$$\mathbf{v}' = \mathbf{K}(\tau)\mathbf{v} + \boldsymbol{\theta}(\mathbf{v},\tau), \tag{3.40}$$

where the matrix $K(\tau)$ has the formal expansion

$$\mathbf{K}(\tau) = \mathbf{K} + \sum_{k=1}^{\infty} \mathbf{K}_k(\tau) e^{-k\beta\tau}$$

with coefficients that are polynomially dependent on τ, and where

$$\boldsymbol{\theta}(\mathbf{v},\tau) = O(\|\mathbf{v}\|^2)$$

is a formal power series in $\mathbf{v}$, $e^{-\beta\tau}$, with coefficients polynomially dependent on τ.

Proof. We recall that we are working under the hypothesis of Theorem 1.1.2, by which the system (3.34) has a formal particular solution in the form of the series (1.16), which after the substitution $\mathbf{x} \mapsto \mathbf{u}, t \mapsto \tau$ passes to the series (3.37). Consequently, (3.38) will be a formal particular solution of system (3.35) if $\mathbf{u_k}(\boldsymbol{\tau}) = \mathbf{x_k}(\ln(\gamma t))$. Therefore the shift $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{v}$ translates system (3.35) into (3.40), which thereby remains asymptotically autonomous and, moreover, formally (i.e. term-by-term) $\boldsymbol{\theta}(\mathbf{v},\tau) \to \boldsymbol{\phi}(\mathbf{v})$ as $\tau \to +\infty$.

The lemma is proved.

We now consider the linear approximating system for (3.40):

$$\mathbf{v}' = \mathbf{K}(\tau)\mathbf{v}. \tag{3.41}$$

Second step. "Autonomization" of the linear system (3.41).

Lemma 3.2.2. *With the aid of the formal linear transformation* $\mathbf{v} = B(\tau)\mathbf{w}$*, where the matrix* $B(\tau)$ *has the asymptotic expansion (3.38), system (3.41) can be translated to the autonomous system*

$$\mathbf{w}' = \mathbf{K}\mathbf{w}.$$

Proof. The matrix $\mathbf{B}(\tau)$ satisfies the following matrix differential equation:

$$\mathbf{B}'(\tau) = \mathbf{K}(\tau)\mathbf{B}(\tau) - \mathbf{B}(\tau)\mathbf{K}.$$

In this last equation we equate matrix coefficients in $e^{-k\beta\tau}$, $k = 0, 1, \ldots$. For $k = 0$ we obtain an obvious identity, and for higher powers $e^{-k\beta\tau}$ we have the matrix equation

$$\mathbf{B}'_{\mathbf{k}}(\tau) = \mathbf{K}\mathbf{B}_{\mathbf{k}}(\tau) - \mathbf{B}_{\mathbf{k}}(\tau)\mathbf{K} + \mathbf{D}_{\mathbf{k}}(\tau),\ \mathbf{D}_{\mathbf{k}}(\tau) = \sum_{j=1}^{k-1} \mathbf{K}_{\mathbf{j}}(\tau)\mathbf{B}_{\mathbf{k-j}}(\tau).$$

Suppose that $\mathbf{B_1}(\tau), \ldots, \mathbf{B_{k-1}}(\tau)$ have already been determined. Then $\mathbf{D_k}$ is a known matrix polynomial in τ and the equation for $\mathbf{B_k}$ can be treated as an inhomogeneous system of n^2 ordinary differential equations with polynomial inhomogeneity. As is known, such a system always has a polynomial particular solution. Thus all the coefficients of the matrix (3.38) can be determined by induction.

The lemma is proved.

After application of the constructed linear transformation to the *nonlinear* system (3.40), that system takes the form

$$\mathbf{w}' = \mathbf{K}\mathbf{w} + \boldsymbol{\omega}(\mathbf{w},\tau). \tag{3.42}$$

Here $\boldsymbol{\omega}(\mathbf{w},\tau) = O(\|\mathbf{w}\|^2)$ is again a formal power series in $\mathbf{w}$ and $e^{-\beta\tau}$ with coefficients that depend polynomially on τ, formally tending to $\boldsymbol{\phi}(\mathbf{w})$ as $\tau \to +\infty$.

Third step. Removal of nonautonomousness in the right sides of system (3.41).

Lemma 3.2.3. *Using the formal transformation*

$$\mathbf{w} = \mathbf{z} + \sum_{m=2}^{\infty} \mathbf{W_m}(\mathbf{z},\tau), \tag{3.43}$$

where $W_m(z,\tau)$ is a homogeneous vector polynomial in z of degree $m = 2,3,\ldots$, whose coefficients are formal power series in $e^{-\beta\tau}$ without free terms and with coefficients that are polynomially dependent on τ, the system (3.42) can be reduced to autonomous form (3.39).

Proof. We first of all note that, since the vector function $\boldsymbol{\omega}(\mathbf{w},\tau)$ tends to $\boldsymbol{\phi}(\mathbf{w})$ as $\tau \to +\infty$, the transformation (3.43) must yield an asymptotic identity, i.e. $\mathbf{W_m}(\mathbf{z},\tau) \to \mathbf{0}$, $m = 2,3,\ldots$ as $\tau \to +\infty$.

The construction of transformation (3.43) will be carried out by induction. Suppose the homogeneous forms $\mathbf{W_2}(\mathbf{z},\tau),\ldots,$
$\mathbf{W_{m-1}}(\mathbf{z},\tau)$ have been found. We consider the so-called homological equation, which the form $\mathbf{W_m}(\mathbf{z},\tau)$ must satisfy:

$$\frac{\partial}{\partial\tau}\mathbf{W_m}(\mathbf{z},\tau) = [\mathbf{W_m}(\mathbf{z},\boldsymbol{\tau}),\mathbf{Kz}] + \mathbf{F_m}(\mathbf{z},\tau), \tag{3.44}$$

where the homogeneous form $\mathbf{F}_m(\mathbf{z},\tau)$ of degree m depends (polynomially) only on forms with smaller indices and where $[\cdot\,,\,\cdot]$ is the vector field commutator.

The form $\mathbf{W}_m(\mathbf{z},\tau)$ enters linearly into the system of partial differential equations (3.44), thereby decomposing $\mathbf{W_m}(\mathbf{z},\tau)$ and $\mathbf{F_m}(\mathbf{z},\tau)$ into power series in $e^{-\beta\tau}$:

$$\mathbf{W_m}(\mathbf{z},\tau) = \sum_{\mathbf{k=1}}^{\infty} \mathbf{W_{m,k}}(\mathbf{z},\tau)e^{-\beta\tau}, \quad \mathbf{F_m}(\mathbf{z},\tau) = \sum_{\mathbf{k=1}}^{\infty} \mathbf{F_{m,k}}(\mathbf{z},\tau)e^{-\beta\tau},$$

where $\mathbf{W_{m,k}}(\mathbf{z},\tau)$ and $\mathbf{F_{m,k}}(\mathbf{z},\tau)$ are homogeneous forms in the variable $\mathbf{z}$ of degree m, whose coefficients are polynomials in τ. Equating like powers of the various exponentials, we obtain

$$\frac{\partial}{\partial\tau}\mathbf{W_{m,k}}(\mathbf{z},\tau) = k\beta\mathbf{W_{m,k}}(\mathbf{z},\tau) + [\mathbf{W_{m,k}}(\mathbf{z},\boldsymbol{\tau}),\mathbf{Kz}] + \mathbf{F_{m,k}}(\mathbf{z},\tau).$$

The space of vector forms in dimension n of degree m is finite dimensional, so that the last system of equations in the partial derivatives of the $\mathbf{W_{m,k}}(\mathbf{z},\tau)$ can be treated as a system of ordinary differential equations with constant coefficients with polynomial inhomogeneity, which always has a polynomial solution.

The lemma is proved.

Successive application of the three steps described above yields the transformation (3.36).

Theorem 3.2.1 is proved.

The idea of “autonomization” of the system (3.35) with the aid of the transformation (3.36), whose coefficients depend not only on powers of $e^{\beta\tau}$ but also on powers of τ, has its origins in perturbation theory. The presence of powers of τ in the expansion (3.36) is analogous to the appearance of so-called secular terms in the standard scheme of perturbation theory (see e.g. [18]). However, in the case considered their negative influence can be mitigated by the presence of negative (i.e., decaying) exponentials, so that the series (3.36) is *possibly* convergent in the case where, naturally, the right side of (3.34) is analytic and the series (1.16) converges, forming an asymptotic solution to system (3.34). Of course this last statement is not an established fact and needs to be put to the test.

Everywhere below we will assume that the matrix $\mathbf{G}$ is diagonal and that its elements are nonnegative rational numbers. So let the linear operator induced by the Kovalevsky matrix have an n_1–dimensional ($n_1 < n$) invariant subspace, to which the restriction of the given operator is nilpotent. We represent the Kovalevsky matrix in the following block form:

$$\mathbf{K} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{J} \end{pmatrix},$$

where $\mathbf{A}$ is an $(n-n_1)\times(n-n_1)$–matrix, $\det \mathbf{A} \neq 0$, $\mathbf{B}$ is an orthogonal $(n-n_1)\times n_1$–matrix, and $\mathbf{J}$ is a singular $n_1 \times n_1$–matrix, specifically a suitable block Jordan matrix with zero diagonal.

We decompose the vector $\mathbf{z}$ into two components: $\mathbf{z} = (\boldsymbol{\xi}, \boldsymbol{\eta})$, where $\boldsymbol{\xi}$ is the projection of $\mathbf{z}$ onto the invariant subspace on which the operator with matrix $\mathbf{K}$ is nonsingular, and $\boldsymbol{\eta}$ is the projection onto the orthogonal complement of that subspace. As was indicated in the preceding section, by using the linear transformation $\boldsymbol{\xi} \mapsto \boldsymbol{\xi} + \mathbf{C}\boldsymbol{\eta}$, the matrix $\mathbf{B}$ can be “killed off”.

We consider the system of differential equations (3.39). This system of equations falls under the aegis of Theorem 3.1.1 of the preceding section, i.e. there exists a formal manifold

$$\boldsymbol{\xi} = \boldsymbol{\varphi}(\boldsymbol{\eta}),$$

on which (3.39) assumes the form

$$\boldsymbol{\eta}' = \mathbf{J}\boldsymbol{\eta} + \ldots = \mathbf{h}(\boldsymbol{\eta}). \tag{3.45}$$

To system (3.45) we can apply the technique that was developed in the first chapter for selecting a quasihomogeneous truncation and constructing a formal asymptotic solution. Let $\mathbf{G_1}$ be some diagonal matrix with rational nonnegative elements. Then we have:

Theorem 3.2.2. *Let system (3.45) be positive semi-quasihomogeneous with respect to the structure induced by the matrix* $\mathbf{G}_1$ *and let* $\boldsymbol{\eta}_0^{\gamma} \in R^{n_1}$ *be some vector so that the identity*

$$-\gamma \mathbf{G_1} \boldsymbol{\eta}_0^{\gamma} = \mathbf{h}_{\mathrm{q}}(\boldsymbol{\eta}_0^{\gamma}) \tag{3.46}$$

holds, where $\gamma = \pm 1$ *is fixed and* $\mathbf{h}_{\mathrm{q}}$ *is a quasihomogeneous truncation of the right side of Eq. (3.45). Then the original system (3.34) has a particular solution* $\mathbf{x}(t)$*, in the form of the following formal series:*

$$\mathbf{x}(t) = \sum_{k=0}^{\infty} \mathbf{x}_k(t)(\gamma t)^{-a_k}, \tag{3.47}$$

whose coefficients $\mathbf{x}_k(t)$ *can in their turn be represented by the series*

$$\mathbf{x}_k(t) = \sum_{j=0}^{\infty} \mathbf{x}_{kj}(t)\,(\ln(\ln(\gamma t))(\ln(\gamma t)))^{-a_{kj}},$$

the coefficients $\mathbf{x}_{kj}(\cdot)$ *being vector polynomials in the iterated exponential* $\ln(\ln(\gamma t))$*, where the sequence of real numbers* $\{a_k\}_{k=0}^{\infty}$*,* $a_0 \geq 0$ *tends monotonically to* $+\infty$ *as* $k \to +\infty$*, and the sequence* $\{a_{kj}\}_{k,j=0}^{\infty}$ *tends monotonically to* $+\infty$*, for each fixed* k *as* $j \to +\infty$*.*

For the indicated particular solution the series (3.47) is an asymptotic series as $t \to \gamma \times \infty$.

In the following section we will give an exact definition of an asymptotic series of type (3.47).

The proof is almost obvious. We use Theorem 3.1.2. From the hypothesis for Theorem 3.2.2—see (3.46)—it follows that system (3.39) has a formal particular solution of the form

$$\mathbf{z}(\tau) = \sum_{k=0}^{\infty} \mathbf{z}_k\,(\ln(\gamma\tau))\,(\gamma\tau)^{-b_k}, \tag{3.48}$$

where the sequence $\{b_k\}_{k=0}^{\infty}$, $b_0 \geq 0$ tends monotonically to $+\infty$ as $k \to +\infty$.

Substituting the series (3.48) into the formula for the transformation $\mathbf{u} \mapsto \mathbf{z}$ of (3.36) and returning to the original variables $\mathbf{x}, \mathbf{t}$, we obtain the assertion of Theorem 3.2.2. Here we are limited to establishing the fact of the existence of a *formal* solution, leaving for the last section the question of the correspondence of this formal solution to an actual solution.

The theorem is proved.

From the construction presented it is evident that if zeros appear among the eigenvalues of the Kovalevsky matrix, calculated for a solution of the type of the quasihomogeneous ray of system (3.45), then in principle it is possible to find a formal particular solution of the original system (3.34) containing iterated logarithms of third order. Repeating the given procedure, it is possible to find a solution containing iterated logarithms of arbitrary order, limited only by the dimension of the system.

In conclusion we remark that, since in the construction of the series (3.47) we used a formal particular solution of system (3.39) that lies on the formal manifold, which generally yields *divergent* power series, so in the typical case the series (3.47) is likewise divergent.

3.3 Systems Implicit with Respect to Higher Derivatives and Kuznetsov's Theory

In this section we will discuss, finally, the correspondence between the formal solutions constructed in the first two sections and certain smooth solutions of systems of differential equations that we have considered. However, we will first study the problem of existence of *formal* asymptotic solutions of yet another class of differential equations. We consider a system of differential equations that is *implicit* with respect to the highest derivative

$$\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}. \tag{3.49}$$

Let the vector function $\mathbf{F}$: $\mathbb{R}^n \to \mathbb{R}^n$ be infinitely differentiable in the neighborhood of $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ and let $\mathbf{x}(t) \equiv \mathbf{0}$ be a particular solution of the system of equations (3.49), where we assume that $\mathbf{F}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

Definition 3.3.1. The system of equations (3.49) is called *quasihomogeneous* with respect to the structure induced by the matrix $\mathbf{G}$ and we will label it, as before, with the index "q",

$$\mathbf{F}_{\mathrm{q}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}, \tag{3.50}$$

if there exists a square matrix $\mathbf{Q}$ with real elements such that for any $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $\mu \in \mathbb{R}^+$ the equality

$$\mathbf{F}_{\mathrm{q}}\left(\mu^{\mathbf{G}}\mathbf{x}, \mu^{\mathbf{G}+\mathbf{E}}\mathbf{p}\right) = \mu^{\mathbf{Q}}\mathbf{F}_{\mathrm{q}}(\mathbf{x}, \mathbf{p}) \tag{3.51}$$

is satisfied.

The equality (3.51) means that the quasihomogeneous system (3.50) is invariant under the action of the group of quasihomogeneous dilations:

$$\mathbf{x} \mapsto \mu^{\mathbf{G}}\mathbf{x}, \quad t \mapsto \mu^{-1}t.$$

Definition 3.3.2. The system of equations (3.49) is called *semi-quasihomogeneous* if the vector function $\mathbf{F}(\mathbf{x}, \mathbf{p})$ can be expanded in a formal series

$$\mathbf{F}(\mathbf{x}, \mathbf{p}) = \sum_{\mathbf{m}=\mathbf{0}} \mathbf{F}_{\mathrm{q}+\chi\mathbf{m}}(\mathbf{x}, \mathbf{p}),$$

such that, for some $\beta \in \mathbb{R} \setminus \{0\}$ and any m, we have the identity

$$\mathbf{F}_{\mathrm{q}+\chi\mathbf{m}}\left(\mu^{\mathbf{G}}\mathbf{x}, \mu^{\mathbf{G}+\mathbf{E}}\mathbf{p}\right) = \mu^{\mathbf{Q}+\mathbf{m}\beta\mathbf{E}}\mathbf{F}_{\mathrm{q}+\chi\mathbf{m}})(\mathbf{x}, \mathbf{p}) \tag{3.52}$$

for $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}}$ and any $\mu \in \mathbb{R}^+$.

If $\beta > 0$, then the system (3.49) is called *positive* semi-quasihomogeneous and *negative* semi-quasihomogeneous in the opposite case ($\beta < 0$).

We will give an interpretation of quasihomogeneity and semi-quasihomogeneity in the language of the Newton diagram and manifold, but waive for the moment the requirement $\mathbf{F}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ and assume only that the components $F^1, \ldots, F^n$ of the vector function $\mathbf{F}(\mathbf{x},\mathbf{p})$ are expanded in formal power series:

$$F^j = \sum_{i_1,\ldots,i_n,l_1,\ldots,l_n} F^j_{i_1,\ldots,i_n,l_1,\ldots,l_n} (x^1)^{i_1} \ldots (x^n)^{i_n} (p^1)^{l_1} \ldots (p^n)^{l_n}. \tag{3.53}$$

Definition 3.3.3. Let

$$F^j_{i_1,\ldots,i_n,l_1,\ldots,l_n} (x^1)^{i_1} \ldots (x^n)^{i_n} (p^1)^{l_1} \ldots (p^n)^{l_n}$$

be some nontrivial monomial of the expansion (3.53) of the j–th component of the vector function $\mathbf{F}(\mathbf{x},\mathbf{p})$. In $\mathbb{R}^{2n}$ we consider a geometrical point with coordinates $(i_1,\ldots,i_n,l_1,\ldots,l_n)$. The set of all such points is called the j-th *Newton diagram* $\mathfrak{D}_j$ of the system of equations (3.49), and its convex hull is called the j–th *Newton polytope* $\mathfrak{P}_j$.

Let $\mathbf{G} = \mathrm{diag}\,(g_1,\ldots,g_n)$ and $\mathbf{Q} = \mathrm{diag}\,(q_1,\ldots,q_n)$ be diagonal matrices with rational elements. It is useful to note that the Newton diagrams $\mathfrak{D}_j$ and the Newton polytopes $\mathfrak{P}_j$ of a quasihomogeneous system of type (3.50) lie on the hyperplanes that are given by the equations

$$g_1 i_1 + \ldots + g_n i_n + (g_1+1) l_1 + \ldots + (g_n+1) l_n = q_j. \tag{3.54}$$

The Newton polytope $\mathfrak{P}_j$ is "based upon" the hyperplanes given by Eq. (3.54) in the case of positive semi-quasihomogeneity, i.e. for any points $(i_1,\ldots,i_n,l_1,\ldots,l_n) \in \mathfrak{P}_j \setminus \Gamma_j$, where Γ_j is a face of the polytope $\mathfrak{P}_j$ lying on the hyperplane considered, we have the following inequality:

$$g_1 i_1 + \ldots + g_n i_n + (g_1+1) l_1 + \ldots + (g_n+1) l_n > q_j. \tag{3.55}$$

In the case of negative semi-quasihomogeneity, $\mathfrak{P}_j$ is "covered" by these hyperplanes, i.e. for any points $(i_1,\ldots,i_n,l_1,\ldots,l_n) \in \mathfrak{P}_j \setminus \Gamma_j$ we have the inequality

$$g_1 i_1 + \ldots + g_n i_n + (g_1+1) l_1 + \ldots + (g_n+1) l_n < q_j. \tag{3.56}$$

Equation (3.54) and likewise inequalities (3.55) and (3.56) can be used for choosing truncations for systems of type (3.49) that are implicit (unsolvable) with respect to the derivative. It is clear that by using the algorithm described in Sect. 3.3 of the preceding chapter, we can represent *any* system (3.27) in semi-quasihomogeneous form. We note that, in selecting a truncation, we can lose *some* or even *all* time derivatives, i.e. the truncated system can be a hybrid system of differential-algebraic equations and even just a system of algebraic equations. We will illustrate this circumstance by an instructive example.

Example 3.3.1. We consider a system of Euler-Poisson equations describing the motion of a massive solid body about a fixed point:

$$\begin{aligned} A\dot{p} + (C - B)qr - Mg(y_0\gamma_z - z_0\gamma_y) = 0, &\quad \dot{\gamma}_x - r\gamma_y + q\gamma_z = 0, \\ B\dot{q} + (A - C)rp - Mg(z_0\gamma_x - x_0\gamma_z) = 0, &\quad \dot{\gamma}_y - p\gamma_z + r\gamma_x = 0, \\ C\dot{r} + (B - A)pq - Mg(x_0\gamma_y - y_0\gamma_x) = 0, &\quad \dot{\gamma}_z - q\gamma_x + p\gamma_y = 0, \end{aligned} \tag{3.57}$$

where (p, q, r) is the projection of the angular velocity vector onto the principal inertial axis of the solid body, $(\gamma_x, \gamma_y, \gamma_z)$ is the projection of a vertical unit vector onto the same axis, $(x_0, y_0, z_0,)$ are the coordinates of the center of mass on that axis, A, B, C are the principal moments of inertia of the solid body, M is its mass, and g is the gravitational constant.

It is easy to see that the system of equations (3.57) is quasihomogeneous with respect to the structure induced by the matrix $\mathbf{G} = \operatorname{diag}(1, 1, 1, 2, 2, 2)$ in the usual sense. The very same quasihomogeneous structure was used by A.M. Lyapunov in his paper [132], dedicated to the study of the branching of solutions of equation (3.57) in the complex time plane.

On the other hand, we can determine a number of other structures with matrix

$$\mathbf{G} = \operatorname{diag}(\mathbf{g_p}, \mathbf{g_q}, \mathbf{g_r}, \mathbf{g_{\gamma_x}}, \mathbf{g_{\gamma_y}}, \mathbf{g_{\gamma_z}}),$$

with respect to which system (3.57) becomes negative semi-quasihomogeneous in the sense of Definition 3.3.2. For instance, the structure with matrix

$$\mathbf{G} = \operatorname{diag}(s, s, s, 2s, 2s, 2s), s \in \mathbb{N}, s \neq 1$$

"cuts" from (3.57) the truncation

$$\begin{aligned} (C - B)qr - Mg(y_0\gamma_z - z_0\gamma_y) = 0, &\quad r\gamma_y - q\gamma_z = 0, \\ (A - C)rp - Mg(z_0\gamma_x - x_0\gamma_z) = 0, &\quad p\gamma_z - r\gamma_x = 0, \\ (B - A)pq - Mg(x_0\gamma_y - y_0\gamma_x) = 0, &\quad q\gamma_x - p\gamma_y = 0, \end{aligned} \tag{3.58}$$

which is an algebraic system of equations.

The quasihomogeneous structure given by the matrix $\mathbf{G} =$ $= \operatorname{diag}(1, 1, 1, 1, 1, 1)$ reduces to a system of differential equations, which in fact describes the motion of a solid under inertia (Euler equations),

$$\begin{aligned} A\dot{p} + (C - B)qr = 0, \dot{\gamma}_x - r\gamma_y + q\gamma_z = 0, \\ B\dot{q} + (A - C)rp = 0, \dot{\gamma}_y - p\gamma_z + r\gamma_x = 0, \\ C\dot{r} + (B - A)pq = 0, \dot{\gamma}_z - q\gamma_x + p\gamma_y = 0, \end{aligned} \tag{3.59}$$

in the context of truncation.

The system of differential-algebraic equations

$$\begin{array}{ll} A\dot{p} + (C - B)qr - Mg(y_0\gamma_z - z_0\gamma_y) = 0, & r\gamma_y - q\gamma_z = 0, \\ B\dot{q} + (A - C)rp + Mgx_0\gamma_z = 0, & \dot{\gamma}_y - p\gamma_z = 0, \\ C\dot{r} + (B - A)pq - Mgx_0\gamma_y = 0, & \dot{\gamma}_z + p\gamma_y = 0, \end{array} \tag{3.60}$$

is obtained as a truncation of system (3.57) in the quasihomogeneous scale generated by the dilation group with matrix $\mathbf{G} = \operatorname{diag}(1, 1, 1, 1, 2, 2)$.

This list of truncations could be extended. Further major possibilities are given by the various particular cases of system (3.57), when two of its three moments of inertia are different or when the center of mass of the body lies on one of the inertial planes.

We consider a truncation of the quasihomogeneous system (3.50). This system admits particular solutions of the quasihomogeneous ray type,

$$\mathbf{x}^{\gamma}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{x}_{\mathbf{0}}^{\gamma}, \tag{3.61}$$

provided that for at least one of $\gamma = \pm 1$ there is a constant nonzero real vector $\mathbf{x}_{\mathbf{0}}^{\gamma}$ such that

$$\mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma\mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma}) = 0. \tag{3.62}$$

We will attempt to find a formal particular solution of system (3.49) in the familiar form

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta.} \tag{3.63}$$

In system (3.49) we make a change of dependent and independent variables

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(\gamma t), \quad s{=}(\gamma t)^{-\beta},$$

after which this system assumes the form

$$\sum_{m=0} s^m F_{\mathrm{q}+\chi m}\left(\mathbf{y}, -\gamma(\beta s\mathbf{y}' + \mathbf{G}\mathbf{y})\right) = 0, \tag{3.64}$$

where the dash denotes differentiation with respect to the new independent variable s, and the formal solution (3.63) becomes

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{x}_k \left(-\frac{\mathbf{1}}{\boldsymbol{\beta}} \ln s\right) s^k. \tag{3.65}$$

Substituting (3.65) into (3.64), we get a chain of equations for $\mathbf{x_k}(\tau)$, $k = 0, 1, 2, \ldots$, where τ is "logarithmic time": $\tau = -\frac{1}{\beta}\ln s = \ln(\gamma t)$.

The constant nonzero vector $\mathbf{x_0} = \mathbf{x}_{\mathbf{0}}^{\gamma}$ can be found from the system (3.62), guaranteeing existence of a particular solution (3.61) of the truncated system (3.50).

For vectors with indices greater than or equal to unity, we have

$$\left[\gamma d_{\mathbf{p}}\mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma \mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma}) \left(\tfrac{\mathbf{d}}{\mathbf{d}\tau} - (\mathbf{G} + \mathbf{k}\boldsymbol{\beta}\mathbf{E})\right) d_x \mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma \mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma})\right] \mathbf{x}_{\mathbf{k}} = \boldsymbol{\Phi}_{\mathbf{k}}(\tau), \tag{3.66}$$

where $\boldsymbol{\Phi}_k(\tau)$ is a polynomial vector function of the preceding coefficients $\mathbf{x}_{\mathbf{0}}(\tau), \ldots, \mathbf{x}_{\mathbf{k-1}}(\tau)$.

If the matrix $d_{\mathbf{p}}\mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma \mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma})$ were nonsingular, then the linear system (3.66) in principle would not be distinguishable from system (1.20) and would be solved in the same way, but the case where $\det(d_{\mathbf{p}}\mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma \mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma})) = \mathbf{0}$ requires careful investigation. According to [83], there exists a symmetric nonsingular matrix $\mathbf{C}$ such that $\mathbf{A} = \gamma \mathbf{C} d_p \mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma \mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma})$ is a symmetric matrix.

We rewrite system (3.66) in the form:

$$\mathbf{A}\mathbf{x}_{\mathbf{k}}' + \mathbf{B}_{\mathbf{k}}\mathbf{x}_{\mathbf{k}} = \boldsymbol{\Psi}_{\mathbf{k}}(\tau), \tag{3.67}$$

where

$$\mathbf{B}_{\mathbf{k}} = \mathbf{C}\left(d_{\mathbf{x}}\mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma \mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma}) - \gamma d_{\mathbf{p}}\mathbf{F}_{\mathrm{q}}(\mathbf{x}_{\mathbf{0}}^{\gamma}, -\gamma \mathbf{G}\mathbf{x}_{\mathbf{0}}^{\gamma})(\mathbf{G} + k\beta\mathbf{E})\right)$$
$$\boldsymbol{\Psi}_{\mathbf{k}} = \mathbf{C}\boldsymbol{\Phi}_{\mathbf{k}}, \quad k = 1, 2, \ldots$$

Since the matrix $\mathbf{A}$ is symmetric, the space $\mathbb{R}^n$ decomposes as the direct sum of range and kernel of the operator with matrix $\mathbf{A}$. Let $\mathbf{x}_{\mathbf{k}} = (\boldsymbol{\xi}_{\mathbf{k}}, \boldsymbol{\eta}_{\mathbf{k}})$, $\boldsymbol{\Psi}_k = (\boldsymbol{\Theta}_k, \boldsymbol{\Omega}_k)$, where $\boldsymbol{\xi}_k, \boldsymbol{\Theta}_k$ and $\boldsymbol{\eta}_k, \boldsymbol{\omega}_k$ are the projections of the vectors $\mathbf{x}_{\mathbf{k}}$ and $\boldsymbol{\Psi}_k$ onto, respectively, the range and kernel of the operator with matrix $\mathbf{A}$. We represent the matrices $\mathbf{A}$ and $\mathbf{B}_{\mathbf{k}}$ in block form

$$\mathbf{A} = \begin{pmatrix} \hat{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_{\mathbf{k}}^{(1)} & \mathbf{B}_{\mathbf{k}}^{(2)} \\ \mathbf{B}_{\mathbf{k}}^{(3)} & \mathbf{B}_{\mathbf{k}}^{(4)} \end{pmatrix}.$$

Here $\mathbf{B}_{\mathbf{k}}^{(1)}, \mathbf{B}_{\mathbf{k}}^{(2)}, \mathbf{B}_{\mathbf{k}}^{(3)}, \mathbf{B}_{\mathbf{k}}^{(4)}$ are the corresponding matrices of dimensions $r \times r$, $r \times (n-r)$, $(n-r) \times r$, $(n-r) \times (n-r)$, where r is the rank of the matrix $\mathbf{A}$.

The system of equations (3.67) is thereby written in the form:

$$\hat{\mathbf{A}}\boldsymbol{\xi}_{\mathbf{k}}' + \mathbf{B}_k^{(1)}\boldsymbol{\xi}_{\mathbf{k}} + \mathbf{B}_{\mathbf{k}}^{(2)}\boldsymbol{\eta}_{\mathbf{k}} = \boldsymbol{\Theta}_{\mathbf{k}}(\tau), \quad \mathbf{B}_{\mathbf{k}}^{(3)}\boldsymbol{\xi}_{\mathbf{k}} + \mathbf{B}_{\mathbf{k}}^{(4)}\boldsymbol{\eta}_{\mathbf{k}} = \boldsymbol{\Omega}_{\mathbf{k}}(\tau). \tag{3.68}$$

We suppose that $\det \mathbf{B}_{\mathbf{k}}^{(4)} \neq \mathbf{0}$. Then from the second system in (3.68) the variables $\boldsymbol{\eta}_k$ can be expressed in terms of $\boldsymbol{\xi}_k$ and the known polynomials $\boldsymbol{\Omega}_k(\tau)$:

$$\boldsymbol{\eta}_k = \left(\mathbf{B}_{\mathbf{k}}^{(4)}\right)^{-1} \left(\boldsymbol{\Omega}_k(\tau) - \mathbf{B}_{\mathbf{k}}^{(3)}\boldsymbol{\xi}_{\mathbf{k}}\right). \tag{3.69}$$

Putting (3.69) into the first system (3.68), we get a system of constant-coefficient differential equations with polynomial inhomogeneity.

$$\hat{\mathbf{A}}\boldsymbol{\xi}'_{\mathbf{k}} + \left(\mathbf{B}^{(1)}_{\mathbf{k}} - \mathbf{B}^{(2)}_{\mathbf{k}}\left(\mathbf{B}^{(4)}_{\mathbf{k}}\right)^{-1}\mathbf{B}^{(3)}_{\mathbf{k}}\right)\boldsymbol{\xi}_{\mathbf{k}} = \\ = \boldsymbol{\Theta}_{\mathbf{k}}(\tau) - \mathbf{B}^{(2)}_{\mathbf{k}}\left(\mathbf{B}^{(4)}_{\mathbf{k}}\right)^{-1}\boldsymbol{\Omega}_{\mathbf{k}}(\tau). \tag{3.70}$$

Since $\det\hat{\mathbf{A}} \neq \mathbf{0}$, system (3.70) always has a polynomial solution $\boldsymbol{\xi}_k(\tau)$. Putting this solution into (3.69), we find a second polynomial $\boldsymbol{\eta}_k(\tau)$.

Thus we can find all the coefficients of the series (3.63) by induction and we have in fact proved

Theorem 3.3.1. *Let system (3.49) be positive quasihomogeneous in the sense of Definition 3.3.2 with respect to the structure with diagonal matrix G, and let its quasihomogeneous truncation (3.50) have a particular solution of (3.62) in the form of a quasihomogeneous ray, i.e. there exist $\gamma = \pm 1$ and a nonzero vector $\mathbf{x}_0^\gamma \in \mathbb{R}^n$ such that the system of equations (3.62) is satisfied. Further let* $\det\mathbf{B}_k^{(4)} \neq 0$ *for all positive integers $k = 1, 2, \ldots$. Then (3.49) has a smooth particular solution $x(t)$, whose dominant term has the asymptotic $\mathbf{x}(t) \sim (\gamma t)^{-G} x_0^\gamma$ as $t \to \gamma \times \infty$.*

Several remarks are in order.

Remark 3.3.1. Again we confine ourselves just to the construction of a formal solution of system (3.49) in the form of a series (3.63). Since by choosing a truncation in the usual way we may lose some differentiability, it is natural to expect that the constructed series will diverge. Below we will formulate a result that allows us to assert that the formal solution constructed indeed corresponds to some actual solution with the required asymptotic. But for its application we will be forced to require that the matrix $\mathbf{G}$ be diagonal and that the power γt in (3.63) be negative, even though in the construction of the *formal* solution this hypothesis was never used.

Theorem 3.3.1 is a refinement of a theorem from the paper [194].

The condition $\det\mathbf{B}^{(4)}_{\mathbf{k}} \neq \mathbf{0}$ is not, of course, a *necessary* condition for the solvability of (3.67). However, in their more general form, such necessary conditions are too awkward and difficult to verify.

We consider a "perturbation" of Example (3.1.1) from Sect. 1.1 of this chapter.

Example 3.3.2. We "spoil" system (3.1) by adding some nonlinear terms to both the first and second equations:

$$\dot{x} + x - y + X(x, y) = 0, \quad \dot{y} + y^2 + Y(x, y) = 0, \tag{3.71}$$

where

$$X(x, y) = \sum_{\substack{i,j=1\\ i+j\geq 2}}^{\infty} X_{ij}x^i y^j, \quad Y(x, y) = \sum_{\substack{i,j=1\\ i+j\geq 3}}^{\infty} Y_{ij}x^i y^j.$$

System (3.71) is positive semi-(quasi)homogeneous with respect to the structure given by the matrix $\mathbf{G} = \mathbf{E} =$diag $(1, 1)$ $(\beta = 1)$. Its truncation has the form

$$y - x = 0, \quad \dot{y} + y^2 = 0.$$

This hybrid system has an obvious asymptotic solution in the form of a ray:

$$x^+(t) = y^+(t) = t^{-1}.$$

We need to look for a particular solution of the full system (3.71) in the form

$$x(t) = \sum_{k=0}^{\infty} x_k(\ln t)t^{-k-1}, \quad y(t) = \sum_{k=0}^{\infty} y_k(\ln t)t^{-k-1}. \tag{3.72}$$

The equations for determining the coefficients $x_k(\tau)$, $y_k(\tau)$ have the form

$$y_k' + (1-k)y_k = \theta_k(\tau), \quad x_k - y_k = \omega_k(\tau),$$

where θ_k, ω_k are polynomials in τ.

This last system is, of course, solvable when $k = 1, 2, \ldots$, where the role of the matrix $\mathbf{B}_\mathbf{k}^{(4)}$ is played by the number $1 \neq 0$.

We consider some systems of *higher order* that are implicit (not solvable) with respect to the highest derivative:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x}^{(\mathbf{p})}), \tag{3.73}$$

for which we can specify conditions for existence of particular solutions in the form of the series (3.63).

We will generally suppose that $\mathbf{x}(t) = \mathbf{0}$ is a solution of system (3.73), i.e. $\mathbf{f}(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}$, and we will seek a solution that is asymptotic to this trivial solution.

Definition 3.3.4. System (3.73) is said to be *quasihomogeneous* with respect to the structure from the matrix $\mathbf{G}$ if its right side $\mathbf{f} = \mathbf{f}_\mathrm{q}$ satisfies the identity

$$\begin{aligned}\mathbf{f}_\mathrm{q}\left(\mu^{\mathbf{G}}\mathbf{x}^{(\mathbf{0})}, \mu^{\mathbf{G}+\mathbf{E}}\mathbf{x}^{(\mathbf{1})}, \mu^{\mathbf{G}+\mathbf{2E}}\mathbf{x}^{(\mathbf{2})}, \ldots, \mu^{\mathbf{G}+\mathbf{pE}}\mathbf{x}^{(\mathbf{p})}\right) = \\ = \mu^{\mathbf{G}+\mathbf{E}}\mathbf{f}_\mathrm{q}\left(\mathbf{x}^{(\mathbf{0})}, \mathbf{x}^{(\mathbf{1})}, \ldots, \mathbf{x}^{(\mathbf{p})}\right)\end{aligned} \tag{3.74}$$

for any $(\mathbf{x}^{(\mathbf{0})}, \mathbf{x}^{(\mathbf{1})}, \mathbf{x}^{(\mathbf{2})}, \ldots, \mathbf{x}^{(\mathbf{p})})$ and $\mu \in \mathbb{R}^+$.

Definition 3.3.5. If there exists $\beta \in \mathbb{R} \setminus \{0\}$ such that the right side of (3.73) can be expanded in a formal series

$$\mathbf{f}(\mathbf{x}^{(\mathbf{0})}, \mathbf{x}^{(\mathbf{1})}, \mathbf{x}^{(\mathbf{2})}, \ldots, \mathbf{x}^{(\mathbf{p})}) = \sum_{m=0} \mathbf{f}_{\mathrm{q}+\chi m}\left(\mathbf{x}^{(\mathbf{0})}, \mathbf{x}^{(\mathbf{1})}, \mathbf{x}^{(\mathbf{2})}, \ldots, \mathbf{x}^{(\mathbf{p})}\right)$$

such that for each m the identity

$$\begin{aligned}\mathbf{f}_{q+\chi m}\left(\mu^{\mathbf{G}}\mathbf{x}^{(0)},\mu^{\mathbf{G+E}}\mathbf{x}^{(1)},\quad \mu^{\mathbf{G+2E}}\mathbf{x}^{(2)},\ldots,\mu^{\mathbf{G+pE}}\mathbf{x}^{(p)}\right) = \\ = \mu^{\mathbf{G+(1+m\beta)E}}\mathbf{f}_{q+\chi m}\left(\mathbf{x}^{(0)},\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(p)}\right)\end{aligned} \tag{3.75}$$

holds for any $\left(\mathbf{x}^{(0)},\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(p)}\right)\in\mathbb{R}^{\mathbf{n(p+1)}}$ and $\mu\in\mathbb{R}^+$, then system (3.73) is said to be *semi-quasihomogeneous.*

We say that system (3.73) is *positive* semi-quasihomogeneous when β is positive, *negative* semi-quasihomogeneous when β is negative.

We consider a particular solution of system (3.73) when the right side of the truncation *does not depend* on derivatives of the vector function $\mathbf{x}(t)$. So, let the truncated system have the form

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}). \tag{3.76}$$

Theorem 3.3.2. *Let system (3.73) be positive semi-quasihomogeneous by Definition 3.3.5 with respect to the structure from the diagonal matrix G, and let its quasihomogeneous truncation (3.76) have a particular solution in the form of a quasihomogeneous ray, i.e. there exist $\gamma=\pm1$ and a nonzero vector x_0^γ satisfying the system of equalities*

$$-\gamma\mathbf{G}\mathbf{x}_0^\gamma = \mathbf{f}_q(\mathbf{x}_0^\gamma). \tag{3.77}$$

Then (3.73) has a smooth particular solution $x(t)$ with smooth principal asymptotic $\mathbf{x}(t)\sim(\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^\gamma)$ as $t\to\gamma\times\infty$.

It should be noted that Theorem 3.3.2 is in a certain sense "more convenient" than Theorem 3.3.1, since its hypothesis does not contain the nonsingularity requirement for the matrix $\mathbf{B}_\mathbf{k}^{(4)}$, whose verification requires additional computation.

Proof. It seems reasonable to reduce the n–dimensional, p–th order system (3.73) to an equivalent np–dimensional, first order system and apply Theorem 3.3.1 to the system so obtained. It is, however, easy to see that the condition $\det\mathbf{B}_\mathbf{k}^{(4)}\neq\mathbf{0}$ will be violated in this case, so that the proof of Theorem 3.3.2 is not immediate. We will find a particular formal solution of (3.73) in the form of a formal series (3.63). As in the proof of the preceding theorem, we will use the substitution

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(\gamma t),\quad \mathbf{s} = (\gamma t)^{-\beta}.$$

The higher derivatives thereby take the form:

$$\mathbf{x}^{(\mathbf{j})}(t) = (\gamma t)^{-(\mathbf{G}+\mathbf{jE})}\mathbf{y}^{[\mathbf{j}]}(s),\quad j=1,\ldots,p,$$

where we have used the notation

$$\mathbf{y}^{[j]}(s) = (-\gamma)^j\sum_{i=0}^{j}\left(C_j^i\left(\prod_{l=0}^{j-i-1}(\mathbf{G}+l\mathbf{E})\right)\sum_{r=0}^{i}s^r a_i^r(\beta)\frac{d^r}{ds^r}\mathbf{y}(s)\right)$$

Here C_j^i is the binomial coefficient $j!/[i!(j-i)!]$ and $a_i^r(\beta)$ is some polynomial in β of degree i with positive integer coefficients. Consequently, by identities (3.74) and (3.75), after the indicated substitutions, the system of equations (3.73) assumes the form

$$-\gamma\beta s\mathbf{y}' = \gamma\mathbf{G}\mathbf{y} + \sum_{\mathbf{m}=\mathbf{0}} s^m \mathbf{f}_{\mathrm{q}+\chi\mathbf{m}}\left(\mathbf{y}, \mathbf{y}^{[\mathbf{1}]}, \ldots, \mathbf{y}^{[\mathbf{p}]}\right). \tag{3.78}$$

Now the search for the coefficients of the series (3.63) will almost literally be a repetition of the search for coefficients of (1.16) in the proof of Theorem 1.1.2. For instance, the zero-th coefficient can be found from system (3.77). From the fact that the right side of the truncated system (3.76) doesn't depend on derivatives, the linear system of equations for determining $\mathbf{x}_k(\tau)$, $\tau = \ln(\gamma t)$, will have the form (1.20), where now $\mathbf{\Phi}_k$ is a polynomial vector function not only the $\mathbf{x_0}, \ldots, \mathbf{x_{k-1}}$, but also of their derivatives with respect to τ. Consequently, $\mathbf{x}_k(\tau)$ can now be found as a polynomial particular solution of this system.

The theorem is proved.

Example 3.3.3. We show how to use Theorem 3.3.2 for finding existence conditions for an asymptotic solution (3.72) of equation (3.71), considered in Example 3.3.2. Differentiating the first equation of the system with respect to time and substituting into it the expression for $\dot{y}$ from the second equation, we get the system

$$\begin{gathered} \dot{x} = -\ddot{x} - \left(y^2 + Y(x,y)\right)\left(1 - \tfrac{\partial X}{\partial y}(x,y)\right) - \tfrac{\partial X}{\partial x}(x,y)\dot{x}, \\ \dot{y} = -y^2 - Y(x,y). \end{gathered} \tag{3.79}$$

In principle, the systems (3.71) and (3.79) are not equivalent, although their solutions that are *asymptotic* to the origin must coincide.

System (3.79) is positive semi-(quasi)homogeneous with respect to the structure from the matrix $\mathbf{G} = \mathbf{E} = \operatorname{diag}(1.1)$. Its truncation clearly has the following form:

$$\dot{x} = -y^2, \quad \dot{y} = -y^2.$$

An asymptotic solution of this system of two first-order equations is obviously

$$x^+(t) = y^+(t) = t^{-1}.$$

Consequently all the hypothesis of Theorem 3.3.2 is fulfilled and system (3.71) has a particular solution for which the series (3.72) represents an asymptotic expansion as $t \to +\infty$.

Remark 3.3.2. Theorem 3.3.2 also remains true in the case where the right sides of system (3.73) are represented by power series containing components of an *infinite* number of derivatives of the vector function $\mathbf{x}(t)$. In Appendix A we will come upon concrete problems where such a situation occurs.

Remark 3.3.3. We should note that Theorems 3.3.1 and 3.3.2 also remain true when the truncated systems have *complex* solutions in the form of a ray. In this case the coefficients of the series (3.63) will likewise be complex.

And thus, finally, we discuss the problem of the correspondence between formal solutions of a system of differential equations and actual solutions with a required asymptotic. For this we consider the general case of a system of type (3.49), implicit in (unsolvable for) its derivatives.

We consider the following construction, due to A.N. Kuznetsov.

We determine recursive iterations of logarithm and exponential:

$$\ln_0 t = e_0(t) = t, \quad \ln_N t = e_{-N}(t) = \ln(\ln_{N-1} t),$$
$$e_N = \ln_{-N} t = \exp\left(e_{N-1}(t)\right), \quad N \in \mathbb{N}.$$

The exponentials with negative indices are associated with logarithms with positive indices and vice versa.

We construct in an inductive manner the following field, $\mathfrak{R}_{M,N}$, $M, N \in \mathbb{Z}$, $M \le N$, of formal functional series by the operations of addition and subtraction of series containing only logarithms with indices that fall in the interval $[M, N]$. The set of real numbers $\mathbb{R}$ contains no logarithms and by definition belongs to the field $\mathfrak{R}_{M,N}$ for all M, N. Further, let $\{a_k\}_{k=0}^{+\infty}$ be a sequence of real numbers that tends strictly monotonically to $+\infty$. If $\{a_k\}_{k=0}^{K}$ is a finite sequence, then we will assume that $a_{K+1} = a_{K+2} = \ldots = +\infty$. Suppose $M < N$ and consider the sequence $\{\hat{x}_k\}_{k=0}^{+\infty}$, of elements that belong to $\mathfrak{R}_{M+1,N} \subset \mathfrak{R}_{M,N}$. Then $\mathfrak{R}_{M,N}$ is a field of series of type:

$$\hat{x} = \sum_{k=0}^{\infty} \hat{x}_k \ln_M^{-a_k} t.$$

On $\mathfrak{R}_{M,N}$ we can define in a natural way, termwise, formal differentiation:

$$\frac{d}{dt}\left(\hat{x}_k \ln_M^{-a_k} t\right) = \left(\frac{d\hat{x}_k}{d\tau} - a_k \hat{x}_k\right) \ln_M^{-a_k} t \prod_{s=0}^{M} \ln_s^{-1} t, \tag{3.80}$$

where $\tau = \ln_{M+1} t$.

We note that any series in $\mathfrak{R}_{M,N}$ can be represented as the sum of monomials with real coefficients. Each monomial is the product of a finite number of iterated logarithms taken to real powers. The totality of all such monomials is totally ordered by the relation $\prec$, which means the following. Let $\phi(t)$, $\psi(t)$ be monomials from the indicated set. Then $\phi \prec \psi$ is equivalent to

$$\lim_{t \to +\infty} \frac{\phi(t)}{\psi(t)} = 0.$$

All series $\hat{x}_k$ that are part of the inductive construction of a given series, and that series itself, will be called *intermediate* series. For any intermediate series we can find the highest logarithm and the highest exponential that appear. Then the series is decomposed by powers of the highest exponential with coefficient series dependent on the lowest exponentials.

Now consider formal vector series $\hat{\mathbf{x}}_{\mathbf{k}}$, whose components belong to some field $\mathfrak{R}_{M,N}$. We consider the system of differential equations (3.49) for which the component vector functions $\mathbf{F}(\mathbf{x}, \mathbf{p})$ are formal power series with respect to the components $\mathbf{x}$ and $\mathbf{p}$. In (3.49) we put, in place of $\mathbf{x}, \dot{\mathbf{x}}$, the formal series $\hat{\mathbf{x}}_{\mathbf{k}}, \widehat{\dot{\mathbf{x}}}_{\mathbf{k}}$. As a result we get a vector series whose components belong to $\mathfrak{R}_{M,N}$. From (3.49), equating to zero the series coefficients by highest exponential, we obtain the *induced relation* for the *intermediate* series. We note that the intermediate series can't converge to zero, i.e. their terms can't converge to zero, but in this case they appear in the induced relation polynomially.

Definition 3.3.6. We say that the series $\hat{x}_k \in \mathfrak{R}_{M,N}$ is the *asymptotic expansion* of the smooth function $x(t)$ $(x(t) \sim \hat{x}_k)$ if, for all $K = 1, 2, \ldots,$ we have

$$x(t) - \sum_{k=0}^{K} \hat{x}_k(t) \ln_M^{-a_k} t = o\left(\ln_M^{-a_K} t\right) \quad \text{as} \quad t \to +\infty.$$

Definition 3.3.7. A *compound asymptotic* $(x(t) \sim \hat{x} \in \mathfrak{R}_{M,N})$ is a recursive relationship between the series $\hat{x}$ and a family of functions $\{x, x_k,\ k \in \mathbb{N} \cup \{0\}\}$ such that these functions are bijectively labeled by the intermediate series $\{\hat{x}, \hat{x}_k, k \in \mathbb{N} \cup \{0\}\}$ and, to the recursive steps for constructing the series $\hat{x}$ of the form

$$\hat{x}_l = \hat{x}_k^* \ln_{M+1}^{-b_k} t,$$

there correspond the asymptotics

$$x_l(t) \sim \sum_{k=0}^{\infty} x_k^* \ln_{M+1}^{-b_k} t \in \mathfrak{R}_{M+1,N}.$$

We will consider only *smooth* asymptotics, i.e. those that undergo differentiation by the rule (3.80).

Definition 3.3.8. Let $\mathbf{x(t)}$ be some smooth solution of system (3.49), whose components have a compound asymptotic. Then we say that the solution itself possesses the compound asymptotic $\mathbf{x} \sim \hat{\mathbf{x}}$. We likewise say that this compound asymptotic satisfies equation (3.49) if all intermediate series satisfy the inductive relationships.

Theorem 3.3.3 ([115]). *If the system (3.49) with analytic vector function* $\mathbf{F}(\mathbf{x}, \mathbf{p})$ *has a formal solution whose components belong to one of the fields* $\mathfrak{R}_{M,N}$*, then this system also has an actual smooth solution with compound asymptotic.*

The proof of this theorem is rather complicated and we won't present it here. It should be noted that, originally, Theorem 3.3.3 was proved for the case where the system was solved for the derivative and the solution was represented as a Laurent type series in fractional negative powers of the independent variable [125]. However, even in this relatively simple case the proof is not at all simple.

This theorem actually fills some lacunae in the proofs of Theorems 3.1.2, 3.1.5, 3.2.2, 3.3.1, and 3.3.2 of this chapter. For the sake of correctness it should be noted that even in its rather general formulation Theorem 3.3.3 does not cover all cases that can arise in constructing formal solutions. In part it can be used only with the analysis of asymptotic solutions, i.e. those *entering* the critical point or *exiting* from it. This means that the given theorem doesn't cover cases of *negative* semi-quasihomogeneous systems in the sense of Definitions 3.3.2 and 3.3.5. There arise likewise a number of other limitations. For instance, in Theorem 3.1.5 we needed to require that the eigenvalues of the linear approximation matrix be real, etc.

We address one concrete problem not directly related to the range of problems considered, but closely connected to them.

Example 3.3.4. For well more than a 100 years the attention of investigators has been attracted by the problem of finding conditions that are sufficient for solutions of the Euler-Poisson equations (3.57) to be single-valued functions of time in the complex domain. This problem had its beginning in a paper of S.V. Kovalevsky [102], published in 1889–1890. In classical cases of the Euler-Lagrange integrability of the Euler-Poisson equations, solutions can be expressed in terms of elliptic functions, which are meromorphic functions of complex time. Kovalevsky studied the question of which cases, beyond those of Euler-Lagrange known at the time, have solutions amenable to expansion into Laurent series

$$\begin{aligned} p(t) &= t^{-g_p} \sum_{k=0}^{\infty} p_k t^k, & \gamma_x(t) &= t^{-g_{\gamma_x}} \sum_{k=0}^{\infty} \gamma_{x_k} t^k, \\ q(t) &= t^{-g_q} \sum_{k=0}^{\infty} q_k t^k, & \gamma_y(t) &= t^{-g_{\gamma_y}} \sum_{k=0}^{\infty} \gamma_{y_k} t^k, \\ r(t) &= t^{-g_r} \sum_{k=0}^{\infty} r_k t^k, & \gamma_z(t) &= t^{-g_{\gamma_z}} \sum_{k=0}^{\infty} \gamma_{z_k} t^k, \end{aligned} \tag{3.81}$$

where among the integers $g_p, g_q, g_r, g_{\gamma_x}, g_{\gamma_y}, g_{\gamma_z}$ at least one is positive, and have no movable singularities other than poles (which in the vast literature published by occidental theoretical physicists, bears the name *Painlevé property*). Also studied was the question as to how many arbitrary constants the coefficients in expansion (3.81) depend. In algebraically integrable cases there must be five. Starting with the equations of motion (3.57), Kovalevsky concluded that the values of the exponents of the maximum powers of t^{-1} in the series (3.81) must be the following:

$$g_p = g_q = g_r = 1, \quad g_{\gamma_x} = g_{\gamma_y} = g_{\gamma_z} = 2.$$

This choice is natural in that the Euler-Lagrange system (3.57) is quasi-homogeneous with respect to the structure from the diagonal matrix $G = \mathrm{diag}\,(1, 1, 1, 2, 2, 2)$. Using the expansions (3.81), Kovalevsky established that the solutions of system (3.57) don't branch if the parameters of the system satisfy relations corresponding to the Euler-Lagrange cases and a new case found by her that presently bears the her name. Next, she found a missing fourth integral of the Euler-Lagrange equations. Kovalevsky's work was subjected to the criticism of A.A. Markov (cf. [70]). In part, he called attention to the fact that the exponents in (3.81) can be other than the ones indicated by Kovalevsky. For example, the choice of $g_p, g_q, g_r, g_{\gamma_x}, g_{\gamma_y}, g_{\gamma_z}$ can be the following:

$$g_p = g_q = g_r = 2, \quad g_{\gamma_x} = g_{\gamma_y} = g_{\gamma_z} = 4.$$

G.G. Appel'rot [2] and P.A. Nekrasov [135] in turn called attention to Markov's result. Appel'rot studied the problem of expansion of solutions in the series (3.81) where

$$g_p = g_q = g_r = s, \quad g_{\gamma_x} = g_{\gamma_y} = g_{\gamma_z} = 2s, \quad s \in \mathbb{N}, \quad s \neq 1.$$

Under corresponding conditions on the parameters of the problem, Appel'rot found Laurent expansions whose coefficients depend on five arbitrary constants, pointing to the possibility of full integrability. These restrictions coincided with the conditions for partial integrability, pointed out earlier by Hess [79]. Nonetheless, the missing first integral still hadn't been found. The problem of finding it was practically settled by Lyapunov [132], who showed that if the parameters $A, B, C, x_0, y_0, z_0, Mg$ are not subjected to the conditions that correspond to the Euler-Lagrange case, then a general solution of the Euler-Poisson system (3.57) branches in the complex domain. Thus in cases other than the three enumerated, only solutions for which the initial conditions are subjected to some previously given relations can be meromorphic functions. This means that Kovalevsky's method is only capable of finding particular cases of integrability. For example, A.A. Bogoyavlensky [20, 21] used Kovalevsky's method to find a family of particular solutions to the Euler- Lagrange equations (3.57), dependent on less than five parameters. This result also raises doubts about results of Appel'rot that have been cited. Below we attempt to give some reasonable explanations for this contradiction from the viewpoint of the theory presented above, which nonetheless cannot be viewed as rigorous mathematical proof. We also will give some bibliographical references. As already noted in the introduction, in the past two decades there has been a growth of interest, especially among theoretical physicists, in the Kovalevsky method as a way of predicting chaotic behavior in dynamical systems. Here we should recall the already cited papers [36, 37, 181]. In these papers the nature of the branching of solutions in the complex domain of solutions of the Lorentz and Henon-Heiles systems is studied. In comparatively recent papers [27, 52] the Duffing-van der Pol equations for the motion of an oscillator were studied from the same viewpoint. Nor was attention lacking for the classical problem of the behavior in the complex plane of the Euler-Poisson system [37]. Further, in the paper [130], the problem of constructing solutions for the system (3.57) in the form (3.81) was fully studied for all possible choices of the numbers $g_p, g_q, g_r, g_{\gamma_x}, g_{\gamma_y}, g_{\gamma_z}$. Conditions were found under which construction of the series (3.81) can be completed and the number of constants on which the coefficients are dependent was determined.

Let some choice of the numbers $g_p, g_q, g_r, g_{\gamma_x}, g_{\gamma_y}, g_{\gamma_z}$ be fixed, determining the lowest terms of the Laurent expansion (3.81). This choice of numbers determines some quasihomogeneous scale. In correspondence with these lowest terms of the expansion there will be a solution of the truncated (in the sense of the scale introduced) equation. For example, the choice of exponents assumed by Appel'rot leads to system (3.58). Taking [37]

$$g_p = g_q = g_r = g_{\gamma_x} = g_{\gamma_y} = g_{\gamma_z} = 1,$$

we obtain the truncated system (3.59). If, as assumed in the same paper [37],

$$g_p = g_q = g_r = g_{\gamma_x} = 1 \quad g_{\gamma_y} = g_{\gamma_z} = 2,$$

the truncation of the system becomes system (3.60).

It should be noted that, from among the systems (3.58)–(3.60), only system (3.59) is a system of differential equations, i.e. only in correspondence with it does the choice of quasihomogeneous scale lead to a regular system. In all remaining cases (except, of course, the systems of exponents studied by Kovalevsky herself) we will be dealing with singular problems. The examples analyzed by us indicate that the series thereby obtained generally diverge and therefore can't represent any meromorphic function. Therefore the analysis of these series has no relation to Kovalevsky's method. Beyond that, formally, we don't even get any guidance from Kovalevsky's method beyond being able to conclude that there is some *infinitely differentiable* solution of system (3.57) for which (3.81) will be the asymptotic series because, with respect to the corresponding scale, (3.57) will be negative semi-quasihomogeneous, and solutions represented by the series (3.81) are not asymptotic. In part, these considerations may explain the fact that the series constructed by Appel'rot contain five arbitrary constants. These series diverge and, evidently, don't correspond to any meromorphic function.

We note that differential equations with polynomial right sides can admit several essentially different families of meromorphic solutions, depending on the "total" number of constant parameters. This phenomenon is easy to explain with the simple example of the Hamiltonian equation

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}; \quad H = \frac{y^2}{2} + f_{n+1}(x), \tag{3.82}$$

where $f = -ax^{n+1} + bx^n + \dots$ is a polynomial with constant coefficients ($a \neq 0$). We consider the problem of the presence for this system of formally meromorphic solutions of the type

$$x = \frac{X_{-\alpha}}{t^{\alpha}} + \frac{X_{-\alpha+1}}{t^{\alpha-1}} + \dots, \quad y = \frac{Y_{-\beta}}{t^{\beta}} + \frac{Y_{-\beta+1}}{t^{\beta-1}} + \dots. \tag{3.83}$$

Here α and β are nonnegative integers such that $\alpha + \beta \geq 1$. The coefficients $X_{-\alpha}, \dots, Y_{-\beta}, \dots$ ($X_{-\alpha} \neq 0, Y_{-\beta} \neq 0$) can take on complex values. We will be interested in the "full" solution of system (3.82). In this case, the coefficients in the expansion (3.83) must contain a "modulus"—an arbitrary parameter. Yet another parameter arises from the replacement in (3.83) of t by $t - t_0$.

The *Kovalevsky number* k of system (3.82) is the number of distinct one-parameter families of meromorphic solutions of the form (3.83). These numbers were introduced in the paper [118] for an arbitrary system with polynomial right sides. For system (3.82), the Kovalevsky number is as follows: if $n = -1, 0, 1$ or $n \geq 4$, then $k = 0$; if $n = 2$, then $k = 1$; finally, for $n = 3$ the number k equals 2. In [118] the Kovalevsky number is calculated for polynomial Hamiltonian systems, describing the dynamics of generalized Toda chains.

In conclusion we consider one more practical problem.

Example 3.3.5. We return to the investigation of the asymptotic solution of the Hill problem, considered in Chap. 1 (see Example 1.4.4). The Hill problem example beautifully illustrates the dependency of the method of constructing asymptotic expansions for particular solutions of systems of ordinary differential equations by choice of a concrete coordinate system. The system of equations (1.69) evidently does not admit any group of dilations whatever with respect to which it would be semi-quasihomogeneous in the classical sense of Definition 1.1.4, apart from those considered. Nor is there any obvious group with respect to which (1.69) would be semi-quasihomogeneous in the sense of Definition 3.3.2 and would satisfy the requirements of Theorem 3.3.1. Nonetheless, in the paper [179] other solutions are constructed that are different from the described classical collision trajectories and that have asymptotic expansions of type (3.63). This fact suggests the existence of some "hidden" quasihomogeneous scale. In fact, if we make the linear change of variables

$$u = p_x + y, \quad v = x + 2p_y, \quad \xi = \frac{2}{3}p_x + \frac{1}{3}y, \quad \eta = x + p_y, \quad \sigma = s, \tag{3.84}$$

Eq. (1.69) assumes the form

$$\begin{aligned} &\dot{u} = v + (v - 2\eta)\sigma^3, && \dot{v} = -u + 2(3\xi - 2u)\sigma^3, \\ &\dot{\xi} = \eta + \tfrac{2}{3}(v - 2\eta)\sigma^3, && \dot{\eta} = (3\xi - 2u)\sigma^3, \\ &\dot{\sigma} = -(3uv - 6\xi v - 4u\eta + 9\xi\eta)\sigma^3. && \end{aligned} \tag{3.85}$$

Rewriting it in the form of a system that is insoluble with respect to its derivatives, system (3.85) is positive semi-quasihomogeneous. Subjecting it to the action of the group of dilations with matrix $G = \text{diag}\,(0, 0, -1, 0, 1)$, i.e. performing the transformation

$$u \mapsto u, \quad v \mapsto v, \quad \xi \mapsto \mu^{-1}\xi, \quad \eta \mapsto \eta, \quad \sigma \mapsto \mu\sigma, \quad t \mapsto \mu^{-1}t,$$

we obtain

$$\begin{aligned}
&\mu\dot{u} - v + \mu^3(v - 2\eta)\sigma^3 = 0,\\
&\mu\dot{v} + u - 2\mu^2(3\xi - 2\mu u)\sigma^3 = 0,\\
&\dot{\xi} - \eta - \tfrac{2}{3}(v - 2\eta)\sigma^3 = 0,\\
&\dot{\eta} - \mu(3\xi - 2\mu u)\sigma^3 = 0,\\
&\dot{\sigma} + 3\xi(3\eta - 2v) + \mu u(3v - 4\eta)\sigma^3 = 0.
\end{aligned} \tag{3.86}$$

Having put $\mu = 0$ in (3.86), we obtain a truncated system that has an obvious one-parameter family of particular solutions

$$u = u_0, \quad v = v_0, \quad \xi = \xi_0 t, \quad \eta = \eta_0, \quad \sigma = \sigma_0 t^{-1},$$

where

$$u_0 = v_0 = 0, \quad \xi_0 = \eta - 0 = c, \quad \sigma_0 = \frac{|c|}{3},$$

$c \in \mathbb{R}$ being a parameter of the family.

Consequently, the system of equations (3.85) has a one-parameter family of particular solutions with asymptotic

$$\begin{aligned}
&u(t) = \sum_{k=1}^{\infty} u_k(\ln t)t^{-k}, && v(t) = \sum_{k=1}^{\infty} v_k(\ln t)t^{-k},\\
&\xi(t) = t\sum_{k=0}^{\infty} \xi_k(\ln t)t^{-k}, && \eta(t) = \sum_{k=0}^{\infty} \eta_k(\ln t)t^{-k},\\
&\sigma(t) = t^{-1}\sum_{k=0}^{\infty} \sigma_k(\ln t)t^{-k}.
\end{aligned} \tag{3.87}$$

From (3.87), by a substitution inverse to (3.84), we obtain that the desired solution of equation (1.68) has a one-parameter family of particular solutions with asymptotic

$$\begin{aligned}
&p_x(t) = t\sum\nolimits_{k=0}^{\infty} p_{x_k}(\ln t)t^{-k}, && x(t) = \sum\nolimits_{k=0}^{\infty} x_k(\ln t)t^{-k},\\
&p_y(t) = \sum\nolimits_{k=0}^{\infty} p_{y_k}(\ln t)t^{-k}, && y(t) = t\sum\nolimits_{k=0}^{\infty} y_k(\ln t)t^{-k},
\end{aligned} \tag{3.88}$$

where

$$p_{x_0} = 3c, \quad x_0 = 2c, \quad p_{y_0} = -c, \quad y_0 = -3c.$$

The analysis carried out in the paper [179] shows that the logarithms in the asymptotics (3.88) are "unkillable". The solutions with asymptotic (3.88) correspond to the so-called nonoscillatory trajectories of the Hill problem, having in configuration space an asymptotic parallel to the Oy axis as $t \to +\infty$. Questions about the qualitative behavior of these trajectories are studied in detail in [179].

Chapter 4
Inversion Problem for the Lagrange Theorem on the Stability of Equilibrium and Related Problems

4.1 On Energy Criteria for Stability

In this chapter we consider problems that we will solve by a method that we originally used for proving instability in cases where linearized equations alone were insufficient. This first section will serve mostly as an introduction. Here we will outline a range of problems and formulate a number of theorems on stability for which converse assertions are introduced in the final sections, their proof being based on the construction of asymptotic solutions. In these theorems the principal stability condition will be the presence of an isolated minimum of some function that plays the role of the potential energy of the system.

So, we first consider a canonical system of Hamiltonian equations

$$\dot{\mathbf{y}} = -d_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) + \mathbf{Q}(\mathbf{x},\mathbf{y}), \quad \dot{\mathbf{x}} = d_y H(\mathbf{x},\mathbf{y}), \quad (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{\mathbf{2n}}, \tag{4.1}$$

describing the motion of a system with total energy

$$H(\mathbf{x},\mathbf{y}) = K(\mathbf{x},\mathbf{y}) + U(\mathbf{x}),$$

where $K(\mathbf{x},\mathbf{y}) = \frac{1}{2}\langle \mathbf{K}(\mathbf{x})\mathbf{y},\mathbf{y}\rangle$ is the kinetic energy of the system and $U(\mathbf{x})$ is the potential under the action of a "perturbing" generalized force $\mathbf{Q}(\mathbf{x},\mathbf{y})$.

Let $\mathbf{x} = \mathbf{0}$ be one of the equilibrium positions of the system, i.e.

$$dU(\mathbf{0}) = \mathbf{0}, \quad \mathbf{Q}(\mathbf{x},\mathbf{0}) \equiv \mathbf{0}.$$

Without loss of generality we will assume that $U(\mathbf{0}) = 0$.

We suppose that the force $\mathbf{Q}(\mathbf{x},\mathbf{y})$ is linear in the generalized momentum: $\mathbf{Q}(\mathbf{x},\mathbf{y}) = \mathbf{R}(\mathbf{x})\mathbf{y}$. Let the system of equations (4.1) be analytic, i.e. the function $U(\mathbf{x})$ and the matrix components $\mathbf{K}(\mathbf{x})$, $\mathbf{R}(\mathbf{x})$ are analytic functions in a neighborhood of $\mathbf{x} = \mathbf{0}$. We can assume that the coordinate system $\{\mathbb{X}\}$ in $\mathbb{R}^n$ is *normal*, i.e. $\mathbf{K}(\mathbf{0}) = \mathbf{E}$, and the matrix $d^2U(\mathbf{0})$ is diagonal.

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Since, in the neighborhood of $\mathbf{x} = \mathbf{0}$, the matrix $\mathbf{K}(\mathbf{x})$ is nonsingular, the generalized force $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ can be represented in the form:

$$\mathbf{Q}(\mathbf{x}, \mathbf{y}) = \mathbf{B}(\mathbf{x}) d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}) = \mathbf{B}(\mathbf{x})\dot{\mathbf{x}},$$

where $\mathbf{B}(\mathbf{x}) = \mathbf{R}(\mathbf{x})\mathbf{K}^{-1}(\mathbf{x})$.

We write the matrix $\mathbf{B}$ as a sum of symmetric and skew-symmetric components:

$$\mathbf{B}(\mathbf{x}) = -\mathbf{D}(\mathbf{x}) + \boldsymbol{\omega}(\mathbf{x}), \quad \mathbf{D}(\mathbf{x}) = -\tfrac{1}{2}\left(\mathbf{B}(\mathbf{x}) + \mathbf{B}^{\mathbf{T}}(\mathbf{x})\right),$$
$$\boldsymbol{\omega}(\mathbf{x}) = \tfrac{1}{2}\left(\mathbf{B}(\mathbf{x}) - \mathbf{B}^{\mathbf{T}}(\mathbf{x})\right),$$

where $(\cdot)^T$ denotes transposition.

We suppose that the system does not have external sources of energy. This means that for any $\mathbf{x}$ the eigenvalues of the matrix $\mathbf{D}(\mathbf{x})$ are nonnegative. In fact, in this case

$$\dot{H}(\mathbf{x}, \mathbf{y}) = -\left(\mathbf{D}(\mathbf{x}) d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}), \mathbf{d}_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})\right) \leq 0,$$

and the energy does not increase under real motions of the system.

The matrix $\mathbf{D}(\mathbf{x})$ is called the matrix of *dissipative forces*, and $\boldsymbol{\omega}(\mathbf{x})$ is the matrix of *gyroscopic forces*. If, furthermore, $\det(\mathbf{D}(\mathbf{x})) \neq 0$ in the neighborhood of equilibrium, then we say that the dissipative forces act on the system with *complete dissipation*. In the absence of gyroscopic forces

$$\mathbf{Q}(\mathbf{x}, \mathbf{y}) = -d_{\dot{\mathbf{x}}} D(\mathbf{x}, \dot{\mathbf{x}}), \quad \dot{\mathbf{x}} = d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}),$$

where

$$D(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\langle \mathbf{D}(\mathbf{x})\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle$$

is the so-called *Rayleigh dissipation function*.

It is clear that the gyroscopic forces do no work. Usually an additional requirement is placed on the matrix of gyroscopic forces, specifically that the differential two-form

$$\boldsymbol{\omega}(\mathbf{x}) = \sum_{\mathbf{i}<\mathbf{j}}^{\mathbf{n}} \omega_{ij}(\mathbf{x})\, dx^i \wedge dx^j,$$

where the $\omega_{ij}(\mathbf{x})$, $i, j = 1, 2, \ldots,$ are the components of the matrix $\boldsymbol{\Omega}(\mathbf{x})$, must be closed (see e.g. [112]), i.e.

$$d\omega = 0,$$

but we will not use this property in the sequel.

We first consider the classical situation where gyroscopic and dissipative forces are absent (system (4.1) is reversible).

Theorem 4.1.1 (Lagrange's theorem). *Let $\Omega(x) \equiv 0$, $D(x) \equiv 0$ and $x = 0$ be a strict local minimum point of the potential energy $U(x)$. Then the trivial solution $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ of system (4.1) is Lyapunov stable.*

On an intuitive physical level the following assertion seems true: if $\mathbf{x} = \mathbf{0}$ isn't a minimum point for $U(\mathbf{x})$, then the equilibrium point considered is unstable. In the analytic case considered this is actually so, but from the *mathematical* point of view this assertion is by no means trivial and, in its most general form, has only been proved quite recently [148]. On the other hand, there exist counterexamples that show that the assertion isn't true even for infinitely differentiable potentials [127,149,196]. For a detailed acquaintance with the converse of Lagrange's theorem we recommend the monograph [154] and also the surveys [96, 156, 160].

But if dissipative and gyroscopic forces are not identically zero, then the following assertion—going back to Kelvin—holds.

Theorem 4.1.2. *Let the point $\mathbf{x} = \mathbf{0}$ be a strict local minimum of the potential energy $U(\mathbf{x})$. Then the trivial solution $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ of system (4.1) is Lyapunov stable.*

In order to prove Theorems 4.1.1 and 4.1.2 it suffices to apply the Lyapunov stability theorem [133], where the Hamiltonian $H(\mathbf{x}, \mathbf{y})$ is used as the Lyapunov function. If dissipative forces are absent, then another known theorem of analytical mechanics reduces to Theorem 4.1.2, i.e. Routh's theorem on the stability of stationary motion.

Theorem 4.1.1 is, of course, a simple particular case of Theorem 4.1.2, but we deliberately distinguish these two assertions, since Theorem 4.1.2—in contrast to its predecessor—*doesn't admit* a converse, even in the analytic case. Even for linear systems, in the absence of dissipative forces, the phenomenon of *gyroscopic stabilization* is possible [40]. In attempting to reverse this assertion, dissipative forces with incomplete dissipation also provide some difficulties. In contrast, the presence of dissipative forces with *complete* dissipation simplifies the situation.

Theorem 4.1.3 ([158]). *Suppose that the equilibrium position of system (4.1) is isolated, i.e. there exists $\varepsilon > 0$ such that*

$$dU(\mathbf{x}) \neq 0 \quad \textit{for any} \quad \mathbf{x} \in \mathbb{R}^{\mathbf{n}}, \quad 0 < \|\mathbf{x}\| < \boldsymbol{\varepsilon},$$

and suppose that dissipation is complete:

$$\langle \mathbf{Q}(\mathbf{x}, \mathbf{y}), d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}) \rangle \leq -C \, \|d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})\|^{2}$$

*for any $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2\mathbf{n}}, 0 < \|\mathbf{x}\| < \boldsymbol{\varepsilon}$, where the constant $C > 0$ doesn't depend on $\mathbf{x}, \mathbf{y}$. If $\mathbf{x} = \mathbf{0}$ is the minimum point for the potential energy $U(\mathbf{x})$, then the solution $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ of system (4.1) is **asymptotically** Lyapunov stable.*

The formulation of the theorem is easily reversed: if $U(\mathbf{x})$ doesn't have a minimum at the coordinate origin $\mathbf{x} = \mathbf{0}$, then instability occurs under the requirements of isolation of the equilibrium position and completeness of dissipation [158]. Both forward and the backward assertions are proved using Krasovskiy's theorem [124], where the Hamiltonian $H(\mathbf{x}, \mathbf{y})$ is used as the auxiliary function.

Closely related to the range of problems considered is the problem on the stability of the critical point for so-called *gradient systems*. Let the non-potential forces $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ be nonzero and further let $\det(\mathbf{B}(\mathbf{x})) \neq 0$. The system of differential equations in $\mathbb{R}^n$

$$\dot{\mathbf{x}} = \mathbf{B}^{-1}(\mathbf{x})\, dU(\mathbf{x}) \tag{4.2}$$

is called *generalized–gradient*.

If $\mathbf{B}(\mathbf{x}) \equiv -\mathbf{D}(\mathbf{x})$, then (4.2) is called *gradient* or *evolutionary*. On the other hand, if $\mathbf{B}(\mathbf{x}) \equiv \boldsymbol{\omega}(\mathbf{x})$, and the form $\boldsymbol{\omega}(\mathbf{x})$ is closed, then (4.2) is Hamiltonian relative to the symplectic structure on $\mathbb{R}^n$ induced by the form $\boldsymbol{\omega}(\mathbf{x})$. In this case the dimension n of the space must of course be even.

Gradient systems were evidently first considered by Lyapunov in connection with the analysis of the stability of critical points of a system of differential equations [133]. They were then studied by Smale in the theory of structural stability [174], and likewise by Thom and his successors in catastrophe theory [67].

Using concepts close to those set forth in [67], system (4.2) can be considered a limiting case of system (4.1) under "small" potential forces.

Along with system (4.2) we likewise consider the reversible Hamiltonian

$$\dot{\mathbf{y}} = -d_{\mathbf{x}} H(\mathbf{x}, \mathbf{y}), \quad \dot{\mathbf{x}} = d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}), \tag{4.3}$$

obtained from (4.1) by discarding nonpotential forces.

System (4.3) is likewise a limiting case of (4.1), but under "large" potential forces.

We will clarify the expressions introduced. Let the Hamiltonian of the system additionally depend on the scalar parameter $c > 0$

$$H(\mathbf{x}, \mathbf{y}, c) = K(\mathbf{x}, \mathbf{y}) + cU(\mathbf{x}).$$

Making the substitution

$$\mathbf{y} \mapsto c\mathbf{y}, \quad \mathbf{t} \mapsto c^{-1}\mathbf{t},$$

in system (4.1) and passing to the limit as $c \to 0$, we obtain a system of differential–algebraic equations

$$\mathbf{0} = -dU(\mathbf{x}) + \mathbf{B}(\mathbf{x}) d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}), \quad \dot{\mathbf{x}} = d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}),$$

equivalent to (4.2).

On the other hand, after the substitution

$$\mathbf{y} \mapsto \sqrt{c}\mathbf{y}, \quad t \mapsto \frac{1}{\sqrt{c}}$$

and passing to the limit $c \to +\infty$, system (4.1) evolves to (4.3).

Thus system (4.1) with nonpotential forces in some sense occupies an intermediate position between the reversible Hamiltonian system (4.3) and the generalized gradient system (4.2). But of course these chosen transitions are purely formal and we can scarcely expect that the solutions of systems (4.1) and (4.2), or (4.1) and (4.3), will be similar to these without the imposition of some additional conditions.

We formulate sufficient conditions for stable equilibrium of system (4.2).

Theorem 4.1.4. *Let* $\mathbf{B}(\mathbf{x}) = -\mathbf{D}(\mathbf{x}) + \boldsymbol{\omega}(\mathbf{x})$ *and suppose for any* x *in some small neighborhood of the origin the bound*

$$\langle \mathbf{D}(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi} \rangle \geq C\,\|\boldsymbol{\xi}\|^2, \quad \boldsymbol{\xi} \in \mathbb{R}^{\mathbf{n}} \tag{4.4}$$

is satisfied, where $C > 0$ *doesn't depend on* $\mathbf{x}$. *If the point* $\mathbf{x} = \mathbf{0}$ *is an isolated minimum of the function* $U(\mathbf{x})$, *then the solution* $\mathbf{x} = \mathbf{0}$ *of system (4.2) is asymptotically Lyapunov stable.*

For the proof, we can apply Lyapunov's theorem on asymptotic stability, where we use the function $U(\mathbf{x})$ as a Lyapunov function [133]. Theorem 4.1.4 admits a converse: if requirement (4.4) is fulfilled and the critical point $\mathbf{x} = \mathbf{0}$ is isolated, then the absence of a minimum of U implies instability. This assertion follows from Lyapunov's instability theorem [133], where the function $U(\mathbf{x})$ again can be used as a Lyapunov function.

The situation changes sharply if the "dissipative component" of the matrix $\mathbf{B}(\mathbf{x})$ is absent and we are dealing with the Hamiltonian system

$$\dot{\mathbf{x}} = \boldsymbol{\Omega}^{-1}(\mathbf{x})\, dU(\mathbf{x}). \tag{4.5}$$

We have

Theorem 4.1.5. *If* $U(\mathbf{x})$ *is a function of fixed sign in the neighborhood of* $\mathbf{x} = \mathbf{0}$ *(i.e. which has neither a strong minimum nor a strong maximum at that point), then the critical point* $\mathbf{x} = \mathbf{0}$ *of system (4.5) is Lyapunov stable.*

This assertion follows from Lyapunov's stability theorem, since the function $U(\mathbf{x})$ is a first integral of system (4.5). But even within the class of linear systems we can construct a counterexample which shows that the converse assertion is false.

In occidental circles it has been quite popular to investigate the stability of equilibrium of system (4.1) *under constantly acting perturbations* [159, 161] (*total stability*). We won't give the corresponding definitions here, but following [159] we introduce the corresponding interpretation. Let the Hamiltonian function and the nonpotential forces in (4.1) depend additionally on a certain set of parameters. It is customary to suppose [161] that the vector of parameters $\mathbf{c}$ belongs to some Banach space, but we limit ourselves to considering the finite case ($\mathbf{c} \in \mathbb{R}^{\mathbf{d}}$). Let the Hamiltonian and nonpotential forces depend analytically on $\mathbf{c}$ in a neighborhood of $\mathbf{c} = \mathbf{0}$, i.e. $U(\mathbf{x}, \mathbf{c})$ and the entries of the matrices $\mathbf{K}(\mathbf{x}, \mathbf{c})$ and $\mathbf{B}(\mathbf{x}, \mathbf{c})$ are analytic functions of $\mathbf{x}, \mathbf{c}$ in some neighborhood of $\mathbf{x} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$. Let changes in the parameters not displace the equilibrium position, i.e. for any $\mathbf{c}$

$$d_{\mathbf{x}} U(\mathbf{0}, \mathbf{c}) \equiv \mathbf{0}.$$

Again, without loss of generality, we will assume that $\mathbf{K}(\mathbf{0}, \mathbf{0}) = \mathbf{E}$.

In this situation the stability, asymptotic stability or instability of system (4.1) under a constantly acting perturbation is in fact equivalent to the stability, asymptotic stability or instability of the critical point $\mathbf{x} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$ of the extended system

$$\begin{gathered} \dot{\mathbf{y}} = -d_{\mathbf{x}} H(\mathbf{x}, \mathbf{y}, \mathbf{c}) + \mathbf{Q}(\mathbf{x}, \mathbf{y}, \mathbf{c}), \quad \dot{\mathbf{x}} = d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}, \mathbf{c}), \\ \dot{\mathbf{c}} = \mathbf{0}, \quad (\mathbf{x}, \mathbf{y}, \mathbf{c}) \in \mathbb{R}^{\mathbf{2n+d}}. \end{gathered} \tag{4.6}$$

Here we won't introduce the theorems formulated in the papers [159, 161], since we will be considering a more complex problem. Clearly, in evolutionary system processes the parameters themselves can change with time, whereby we make the natural assumption that the rate of change of the parameters depends both of the values of the parameters themselves at a given moment of time and on the values of the phase variables, i.e. instead of system (4.6) we consider the more general system

$$\begin{gathered} \dot{\mathbf{y}} = -d_{\mathbf{x}} H(\mathbf{x}, \mathbf{y}, \mathbf{c}) + \mathbf{Q}(\mathbf{x}, \mathbf{y}, \mathbf{c}), \quad \dot{\mathbf{x}} = d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y}, \mathbf{c}), \\ \dot{\mathbf{c}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{c}) \end{gathered} \tag{4.7}$$

and suppose that if the initial values of the parameters are zero and the system is in equilibrium, then the parameters won't change further with time, i.e. $\mathbf{f}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$. It turns out that we can draw conclusions about the stability or instability of solutions of the system (4.7) based only on the presence of a minimum of the first nontrivial form in the expansion of potential energy at the zero values of the parameters.

We have

Theorem 4.1.6. *Let the following conditions be satisfied:*

1. $B(x, c) = -D(x, c) + \omega(x, c)$ and for any $\xi \in R^n$

$$\langle \mathbf{D}(\mathbf{0}, \mathbf{0}) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \geq C_1 \|\boldsymbol{\xi}\|^{\mathbf{2}}, \quad C_1 > \mathbf{0}, \tag{4.8}$$

2. Let $\Lambda = d_{\mathbf{c}} f(0, 0, 0)$ and for any $\eta \in R^d$

$$\langle \Lambda \boldsymbol{\eta}, \boldsymbol{\eta} \rangle \leq -C_2 \|\boldsymbol{\eta}\|^2, \quad C_2 > 0. \tag{4.9}$$

Consider the expansion of the potential function $U(x, c)$ into a Maclaurin series in the neighborhood of $x = 0$

$$U(\mathbf{x}, \mathbf{c}) = \sum_{\mathbf{m}=\mathbf{M}}^{\infty} U_m(\mathbf{x}, \mathbf{c}), \quad M \geq 2,$$

(i.e. several of the first forms in the expansion may turn out to be identically equal to zero for any c). If the form $U_M(\mathbf{x}, \mathbf{0})$ has an isolated local minimum at the point $\mathbf{x} = \mathbf{0}$, then the equilibrium position $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$ of system (4.7) will be asymptotically stable. Conversely, if the function $U_M(\mathbf{x}, \mathbf{0})$ does not have a local minimum at the isolated *critical point $\mathbf{x} = \mathbf{0}$, then this equilibrium position is unstable.*

Here we depart from our rules for presenting the material of this section. First, we haven't proved the formulated assertions, either because they are contained in one or another monograph on the theory of stability of motion or because they are trivial consequences of known theorems. Theorem 4.1.6 is new and hence we give its full proof. Secondly, we have not yet rigorously formulated the assertions on instability, leaving their consideration for the ensuing section. However, the proof of asymptotic stability and instability in Theorem 4.1.6 was based on a single technique, so that it would have been illogical to break it into fragments. In Sect. 4.3 we will give a different proof of the circumstance of instability, based on the presence of asymptotic trajectories in the system.

Proof of Theorem 4.1.6. We consider first the simpler case $M = 2$. The first approximation system has the form

$$\dot{\mathbf{y}} = -\mathbf{A}\mathbf{x} - (\mathbf{D}-\boldsymbol{\omega})\mathbf{y}, \quad \dot{\mathbf{x}} = \mathbf{y}, \quad \dot{\mathbf{c}} = \boldsymbol{\Lambda}\mathbf{c}+\boldsymbol{\Lambda}_{\mathbf{x}}\mathbf{x}+\boldsymbol{\Lambda}_{\mathbf{y}}\mathbf{y}, \tag{4.10}$$

where $\boldsymbol{\Lambda}_{\mathbf{x}}$, $\boldsymbol{\Lambda}_{\mathbf{y}}$ are some $d \times n$–matrices, the matrices $\mathbf{D}(\mathbf{0}, \mathbf{0})$, $\boldsymbol{\Omega}(\mathbf{0}, \mathbf{0})$ are denoted by $\mathbf{D}$, $\boldsymbol{\Omega}$, and $\mathbf{A} = d^2U(\mathbf{0}, \mathbf{0})$.

The characteristic equation for Eq. (4.10) has the form

$$\det\left(\lambda^2\mathbf{E_n}+\lambda(\mathbf{D}-\boldsymbol{\Omega}) + \mathbf{A}\right)\det(\boldsymbol{\Lambda} - \lambda\mathbf{E_d}) = 0,$$

where $\mathbf{E_n}$, $\mathbf{E_d}$ are identity matrices of orders n and d, respectively.

First note that, by (4.9), all roots of the equation

$$\det(\Lambda - \lambda\mathbf{E_d}) = 0$$

have negative real part.

Furthermore, the first two equations in (4.10)s are distinct. The characteristic equation of the resulting systems is

$$\det\left(\lambda^2\mathbf{E_n} + \boldsymbol{\lambda}(\mathbf{D}-\boldsymbol{\Omega}) + \mathbf{A}\right) = 0. \tag{4.11}$$

Since, by the hypothesis of the theorem, $\mathbf{x} = \mathbf{0}$ is the unique critical point of the quadratic form

$$U_2(\mathbf{x}, \mathbf{0}) = \frac{1}{2}\langle\mathbf{A}\mathbf{x}, \mathbf{x}\rangle,$$

and the dissipation is total (see (4.8)), we have by Salvadori's theorem [158] that the considered equilibrium position of the linear system is asymptotically stable if the form $U_2(\mathbf{x}, \mathbf{0})$ is positive definite, and is unstable if this form takes on negative values. But asymptotic stability in a linear system can only be exponential. Therefore all roots of (4.11) have negative real parts if $U_2(\mathbf{x}, \mathbf{0})$ has an isolated minimum at the origin. But instability in a linear system can only arise either by a root of the characteristic equation with positive real part, by a multiple zero or pure imaginary root. Let $\lambda = \pm i\theta$ be two pure imaginary complex conjugate roots of (4.11). Then there exist two real vectors $\mathbf{a}, \mathbf{b}$, $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \neq \mathbf{0}$ such that

$$-\theta^2\mathbf{a} - \theta(\mathbf{D} - \boldsymbol{\Omega})\mathbf{b} + \mathbf{A}\mathbf{a} = \mathbf{0}, \quad -\theta^2\mathbf{b} + \theta(\mathbf{D} - \boldsymbol{\Omega})\mathbf{a} + \mathbf{A}\mathbf{b} = \mathbf{0}.$$

Multiplying the first scalar equation by $\mathbf{b}$, the second by $\mathbf{a}$ and subtracting the first from the second, we obtain

$$\theta\left(\langle \mathbf{D}\mathbf{a}, \mathbf{a}\rangle + \langle \mathbf{D}\mathbf{b}, \mathbf{b}\rangle\right) = 0,$$

from which it follows that $\theta = 0$.

However, since $\det \mathbf{A} \neq 0$, Eq. (4.11) doesn't have any zero roots, and instability can only be achieved by the presence of a root with a positive real part. To complete the proof of the theorem in the case $M = 2$, it remains to apply the Lyapunov theorems on asymptotic stability and instability by first approximation.

The proof of Theorem 4.1.6 for the case $M > 2$ is based on the center manifold theorem [137]. In this case the characteristic equation of the first approximation system (4.10) has $n + d$ roots with negative real part and n zeros. Setting $\mathbf{B} = \mathbf{B}(\mathbf{0}, \mathbf{0})$ and making the linear substitution

$$\mathbf{x} = \mathbf{u} + \mathbf{B}^{-1}\mathbf{v}, \quad \mathbf{y} = \mathbf{v}, \quad \mathbf{c} = \boldsymbol{\sigma},$$

we obtain the system of differential equations

$$\begin{aligned}
\dot{\mathbf{v}} &= -d_{\mathbf{x}}H(\mathbf{u} + \mathbf{B}^{-1}\mathbf{v}, \mathbf{v}, \boldsymbol{\sigma}) + \mathbf{Q}(\mathbf{u} + \mathbf{B}^{-1}\mathbf{v}, \mathbf{v}, \boldsymbol{\sigma}),\\
\dot{\mathbf{u}} &= d_{\mathbf{y}}H(\mathbf{u} + \mathbf{B}^{-1}\mathbf{v}, \mathbf{v}, \boldsymbol{\sigma}) + \mathbf{B}^{-1}\big(d_{\mathbf{x}}H(\mathbf{u} + \mathbf{B}^{-1}\mathbf{v}, \mathbf{v}, \boldsymbol{\sigma}) - \\
&\qquad - \mathbf{Q}(\mathbf{u} + \mathbf{B}^{-1}\mathbf{v}, \mathbf{v}, \boldsymbol{\sigma})\big)\\
\dot{\boldsymbol{\sigma}} &= \mathbf{f}(\mathbf{u} + \mathbf{B}^{-1}\mathbf{v}, \mathbf{v}, \boldsymbol{\sigma}).
\end{aligned} \tag{4.12}$$

The first approximation system for (4.12) now has the form

$$\dot{\mathbf{v}} = \mathbf{B}\mathbf{v}, \quad \dot{\mathbf{u}} = \mathbf{0}, \quad \dot{\boldsymbol{\sigma}} = \boldsymbol{\Lambda}\boldsymbol{\sigma} + \boldsymbol{\Lambda}_{\mathbf{x}}u + (\boldsymbol{\Lambda}_y + \boldsymbol{\Lambda}_{\mathbf{x}}\mathbf{B}^{-1})\mathbf{v}.$$

We make yet another linear change of variables

$$\boldsymbol{\xi} = \mathbf{v}, \quad \boldsymbol{\eta} = \boldsymbol{\sigma} + \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathbf{x}}\mathbf{u}, \quad \boldsymbol{\zeta} = \mathbf{u}.$$

System (4.12) now acquires the form

$$
\begin{aligned}
\dot{\boldsymbol{\xi}} &= -d_{\mathrm{x}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) + \\
&\quad + \mathbf{Q}(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\zeta}, \boldsymbol{\zeta}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}), \\
\dot{\boldsymbol{\eta}} &= \mathbf{f}(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) + \\
&\quad + \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}} \big(d_{\mathrm{y}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) + \\
&\quad + \mathbf{B}^{-1} \big(d_{\mathrm{x}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) - \\
&\quad - \mathbf{Q}(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) \big) \big), \\
\dot{\boldsymbol{\zeta}} &= d_{\mathrm{y}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) + \\
&\quad + \mathbf{B}^{-1} \big(d_{\mathrm{x}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) - \\
&\quad - \mathbf{Q}(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\eta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) \big).
\end{aligned}
\tag{4.13}
$$

The first approximation system for (4.13) is yet further simplified:

$$
\dot{\boldsymbol{\xi}} = \mathbf{B}\boldsymbol{\xi}, \quad \dot{\boldsymbol{\eta}} = \boldsymbol{\Lambda}\boldsymbol{\eta} + (\boldsymbol{\Lambda}_{\mathrm{y}} + \boldsymbol{\Lambda}_{\mathrm{x}}\mathbf{B}^{-1})\boldsymbol{\xi}, \quad \dot{\boldsymbol{\zeta}} = \mathbf{0}.
$$

Consequently, we can apply the center manifold theorem [137] to system (4.1). This manifold will be "real", and not formal, as in Theorem 3.1.1 (although it might have only finite smoothness), since all the nonzero roots of the first approximation system will have negative real parts. It will, moreover, be exponentially attracting. The variables $\boldsymbol{\zeta}$ are critical, so that the center manifold will have the form

$$
\boldsymbol{\xi} = \boldsymbol{\varphi}(\boldsymbol{\zeta}), \quad \boldsymbol{\eta} = \boldsymbol{\psi}(\boldsymbol{\zeta}),
$$

where the vector functions $\boldsymbol{\varphi}(\boldsymbol{\zeta}), \boldsymbol{\psi}(\boldsymbol{\zeta})$ can be expanded in formal Maclaurin series in a neighborhood of the point $\boldsymbol{\varsigma} = \mathbf{0}$, where $d\boldsymbol{\varphi}(\mathbf{0}) = \mathbf{0}$, $d\boldsymbol{\psi}(\mathbf{0}) = \mathbf{0}$.

The system, reduced on the center manifold, has the form

$$
\dot{\boldsymbol{\varsigma}} = \mathbf{Z}(\boldsymbol{\varsigma})
$$

where the matrix $d\mathbf{Z}(\mathbf{0})$ is nilpotent.

The vector functions $\boldsymbol{\varphi}(\boldsymbol{\zeta}), \boldsymbol{\psi}(\boldsymbol{\zeta})$ satisfy a rather complicated system of first order partial differential equations. We write the portion of it in which the partial derivatives of the vector function $\boldsymbol{\varphi}(\boldsymbol{\zeta})$ appear:

$$
\begin{aligned}
& -d_{\mathrm{x}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\varphi}, \boldsymbol{\varphi}, \boldsymbol{\psi} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) + \\
& + \mathbf{Q}(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\varphi}, \boldsymbol{\varphi}, \boldsymbol{\psi} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) = \\
& = d\boldsymbol{\varphi} \big\{ d_{\mathrm{y}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\varphi}, \boldsymbol{\varphi}, \boldsymbol{\psi} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) + \\
& + \mathbf{B}^{-1} \big(d_{\mathrm{x}} H(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\varphi}, \boldsymbol{\varphi}, \boldsymbol{\psi} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) - \\
& - \mathbf{Q}(\boldsymbol{\zeta} + \mathbf{B}^{-1}\boldsymbol{\varphi}, \boldsymbol{\varphi}, \boldsymbol{\psi} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathrm{x}}\boldsymbol{\zeta}) \big) \big\}.
\end{aligned}
\tag{4.14}
$$

From (4.14) it is clear that the expansion of $\boldsymbol{\varphi}(\boldsymbol{\zeta})$ into a formal Maclaurin series begins with terms of order $M-1$. Moreover, the equation defining the form $\boldsymbol{\varphi}_{M-1}(\zeta)$ has the appearance

$$-d_{\varsigma} U_M(\varsigma, 0) + \mathbf{B}\boldsymbol{\varphi}_{\mathbf{M-1}}(\boldsymbol{\zeta}) = \mathbf{0}.$$

Developing the first nontrivial forms in the expansion of the vector field $\mathbf{Z}(\varsigma)$ into a Maclaurin series in the neighborhood of $\varsigma = 0$ and setting $\boldsymbol{\xi} = \boldsymbol{\varphi}(\boldsymbol{\zeta})$, $\boldsymbol{\eta} = \boldsymbol{\psi}(\boldsymbol{\zeta})$ in the last group of equations of (4.13), we find that this vector field is "almost" generalized–gradient:

$$\dot{\varsigma} = \mathbf{B}^{-1} d_{\varsigma} U_M(\varsigma, \mathbf{0}) + \tilde{\mathbf{Z}}(\varsigma) \tag{4.15}$$

$\tilde{\mathbf{Z}}(\varsigma) = o\left(\|\varsigma\|^{\mathbf{M-1}}\right)$ as $\varsigma \to \mathbf{0}$.

Since the invariant center manifold is attractive, the asymptotic stability (resp. instability) of the original system is equivalent to the asymptotic stability (resp. instability) of system (4.15). If $\tilde{\mathbf{Z}}(\varsigma) \equiv \mathbf{0}$, then the necessity of the condition results from the asymptotic stability (resp. instability) of the model system

$$\dot{\varsigma} = \mathbf{B}^{-1}\, d_{\varsigma} \mathbf{U}_{\mathbf{M}}(\varsigma, \mathbf{0}). \tag{4.16}$$

There is the question as to whether the properties of asymptotic stability and instability for system (4.16) are structurally stable. As follows from the general theory [100], the property of asymptotic stability for small perturbations of homogeneous systems is in fact structural, and instability too is structurally stable if, in the truncated system, it arises from the presence of linear increasing solutions of the ray type. In the case considered the situation is simpler still, since the model system (4.16) is generalized-gradient and the Lyapunov function for it, and consequently for the full system (4.15), can be written explicitly. Since $\varsigma = \mathbf{0}$ is an isolated critical point of the form $U_M(\varsigma, \mathbf{0})$, we have

$$\|d_{zt} U_M(\varsigma, \mathbf{0})\| \geq C_{\mathbf{3}} \|\varsigma\|^{\mathbf{M-1}}, \quad C_{\mathbf{3}} > 0.$$

We estimate the time derivative of the function $U_M(\varsigma, \mathbf{0})$ by means of system (4.15):

$$U_M(\varsigma, \mathbf{0}) = \langle d_{\varsigma} U_M(\varsigma, \mathbf{0}), \mathbf{B}^{-1} \mathbf{d}_{\varsigma} U_M(\varsigma, \mathbf{0}) + \tilde{\mathbf{Z}}(\varsigma) \rangle \geq$$
$$\geq C_4 \|\varsigma\|^{\mathbf{2M-2}} + o(\|\varsigma\|^{\mathbf{2M-2}}), \quad C_4 > 0.$$

Thus in a small neighborhood of $\varsigma = \mathbf{0}$ the function $U_M(\varsigma, \mathbf{0})$ is positive definite and the form $U_M(\varsigma, \mathbf{0})$ can be used as a Lyapunov function for system (4.15). This system is asymptotically stable if $U_M(\varsigma, \mathbf{0})$ has an isolated minimum, and unstable if $U_M(\varsigma, \mathbf{0})$ does not have a minimum at the point $\varsigma = \mathbf{0}$, which by itself implies asymptotic stability (resp. instability) of the equilibrium position $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$ of system (4.7).

The theorem is proved.

We can give the hypothesis of the theorem just proved the following interpretation. System (4.7) consists of two interdependent systems, one of which is described by the change of the phase variables $(\mathbf{x}, \mathbf{y})$, and the other is described by the change in the parameter vector $\mathbf{c}$. For asymptotic stability it is necessary that the first subsystem be asymptotically stable for fixed values of the parameter $\mathbf{c}$ that differ but little from $\mathbf{c} = \mathbf{0}$. If we fix some small nonzero values of the phase variables $(\mathbf{x}, \mathbf{y})$ then, by applying the explicit function theorem, it is possible to show that the equilibrium position $\mathbf{c} = \mathbf{0}$ of the second subsystem doesn't disappear, but is "displaced" and remains asymptotically stable. These properties are *structurally stable,* whereas simple stability of disconnected systems for fixed values of the parameter or phase variables does *not guarantee* stability for connected systems. We illustrate this with a simple example.

Example 4.1.1. We consider the system of equations

$$\dot{y} = -\omega^2(1 + c_1)x, \quad \dot{x} = y, \quad \dot{c}_1 = -c_2, \quad \dot{c}_2 = c_1, \quad \omega \neq 0. \tag{4.17}$$

The equilibrium position $x = y = 0$ for a Hamiltonian system with Hamiltonian

$$H(x, y, c_1, c_2) = \frac{y^2}{2} + \omega^2(1 + c_1)\frac{x^2}{2}$$

is stable for any choice of the parameters $-1 < c_1 < +\infty$, $-\infty c_2 < +\infty$. The equilibrium position $c_1 = c_2$ of the system

$$\dot{c}_1 = -c_2, \quad \dot{c}_2 = c_2,$$

describing a change of parameters, is likewise stable.

However, for $\omega = \frac{k}{2}$, $k \in \mathbb{Z}$, the equilibrium position $x = y = c_1 = c_2 = 0$ of system (4.17) is unstable. In fact, (4.17) can be rewritten as a second order scalar equation

$$\ddot{x} + \omega^2\left(1 + \varepsilon\cos(t - t_0)\right)x = 0$$

where

$$\varepsilon = \sqrt{c_1^2(t_0) + c_2^2(t_0)}.$$

In this equation, for values of frequencies that are close to half-integers, the phenomenon of parametric resonance [18] occurs, which leads to instability.

It is evident that the hypothesis of Theorem 4.1.6 can't, in general, be substantially weakened, in part because it requires that the presence or absence of a minimum of $U(\mathbf{x}, \mathbf{c})$ be seen in the *first* nontrivial form in the expansion, a requirement that seems rather stringent when we consider the essence of the situation. The point is that asymptotic stability or instability of system (4.7) is determined by the asymptotic stability or instability of the system reduced on the center manifold. This system is generalized-gradient only in the first nontrivial approximate. If asymptotic stability or instability does not occur, then the group of equations that describe changes in the parameter can exert a substantial influence on the subsequent terms, and in this case the *energy* criterion for stability can scarcely hold.

We consider, finally, the motion of a conservative system restricted by $d < n$ additional *nonholonomic constraints*

$$\langle \mathbf{w}^p(\mathbf{x}), \dot{\mathbf{x}} \rangle = \sum_{j=1}^{n} w_j^p(\mathbf{x})\dot{\mathbf{x}}^{\mathbf{j}} = \mathbf{0}, \quad p = 1, \ldots, d. \tag{4.18}$$

We consider an $n \times d$ matrix $\mathbf{W}(\mathbf{x})$, whose columns

$$\mathbf{w}^{\mathbf{1}}(\mathbf{x}), \ldots, \mathbf{w}^{\mathbf{d}}(\mathbf{x})$$

are covectors of the holonomic constraints. The constraint equations (4.18) can then be written in vector–matrix form

$$\mathbf{W}^{\mathbf{T}}(\mathbf{x})\dot{\mathbf{x}} = \mathbf{0}$$

or

$$\mathbf{W}^{\mathbf{T}}(\mathbf{y})\, d_{\mathbf{y}}\mathbf{H}(\mathbf{x}, \mathbf{y}) = \mathbf{0}. \tag{4.19}$$

We will assume that the components of the matrix $\mathbf{W}(\mathbf{x})$ are analytic in the neighborhood of $\mathbf{x} = \mathbf{0}$.

The motion of such a mechanical system is described by a system of differential–algebraic equations

$$\begin{gathered} \dot{\mathbf{y}} = -d_{\mathbf{x}}H(\mathbf{x}, \mathbf{y}) + \mathbf{W}(\mathbf{x})\boldsymbol{\lambda}, \quad \dot{\mathbf{x}} = d_{\mathbf{y}}H(\mathbf{x}, \mathbf{y}), \\ \mathbf{W}^{\mathbf{T}}(\mathbf{x})\, d_{\mathbf{y}}H(\mathbf{x}, \mathbf{y}) = \mathbf{0}, \end{gathered} \tag{4.20}$$

where the covector $\boldsymbol{\lambda} \in \mathbb{R}^d$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$, is a set of Lagrange multipliers.

As before, let $\mathbf{x} = \mathbf{0}$ be a critical point of the potential energy $U(\mathbf{x})$. Then the triple $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, $\boldsymbol{\lambda} = \mathbf{0}$ is an equilibrium solution of system (4.20). In general, not only critical points of the potential energy $U(\mathbf{x})$ are equilibrium positions: the equations for finding the equilibrium positions has the form

$$-dU(\mathbf{x}) + \mathbf{W}(\mathbf{x})\boldsymbol{\lambda} = \mathbf{0}.$$

The equilibrium positions that are determined by critical points of the potential energy (with $\boldsymbol{\lambda} = \mathbf{0}$), are called *equilibria of the first kind*, and all others (here $\boldsymbol{\lambda} \neq \mathbf{0}$) are called *equilibria of the second kind*. If $\det\big(d^2U(\mathbf{0})\big) \neq 0$ and the matrix $\mathbf{W}(\mathbf{0})$ has maximum rank, then the equilibrium position of the first kind $\mathbf{x} = \mathbf{0}$ lies on the smooth regular d–dimensional surface consisting of the equilibrium positions of the system. All equilibria from a small punctured neighborhood of zero are equilibria of the second kind [110]. Further information concerning geometric equilibria of nonautonomous systems can be obtained from the cited paper [110].

None of the energy criteria is generally applicable to equilibria of the second kind and so we will deal only with equilibria of the first kind.

To begin, we note that if in some neighborhood of the point $\mathbf{x} = \mathbf{0}$ the matrix $\mathbf{W}(\mathbf{x})$ has maximal rank then, by elimination of Lagrange multipliers $\boldsymbol{\lambda}$, Eq. (4.20) can again be reduced to the form (4.1), where the nonpotential force $\mathbf{Q}(\mathbf{x}, \mathbf{y}) = \mathbf{W}(\mathbf{x})\boldsymbol{\lambda}$ controls the motion on the constraints, provided only that the initial conditions satisfy the constraint equations. This means that the actual motions of the system lie on the invariant manifold given by (4.19).

In fact, differentiating (4.19) with respect to time and substituting the first and second groupings of equations of the system (4.20) into the relations obtained, we have

$$\mathbf{W}^{\mathbf{T}}(\mathbf{x})\, d^{\mathbf{2}}_{\mathbf{yy}} H(\mathbf{x}, \mathbf{y})\mathbf{W}(\mathbf{x})\boldsymbol{\lambda} = \mathbf{F}(\mathbf{x}, \mathbf{y}), \tag{4.21}$$

where

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = -\boldsymbol{\Gamma}\left(\mathbf{x}, d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})\right) + \\ + \mathbf{W}^{\mathbf{T}}(\mathbf{x})\left(d^{\mathbf{2}}_{\mathbf{yy}} H(\mathbf{x}, \mathbf{y}) d_{\mathbf{x}} H(\mathbf{x}, \mathbf{y}) - d^{\mathbf{2}}_{\mathbf{yx}} H(\mathbf{x}, \mathbf{y}) d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})\right),$$

and $\boldsymbol{\Gamma}(\mathbf{x}, \cdot)$ is the transformation, quadratic in its second argument, whose coefficients are the partial derivatives of the components of the matrix $\mathbf{W}^{\mathbf{T}}(\mathbf{x})$.

Since the Hessian is $d^2_{\mathbf{yy}} H(\mathbf{x}, \mathbf{y}) = \mathbf{K}(\mathbf{x})$ and $\mathbf{K}(\mathbf{0}) = \mathbf{E}$, we have

$$\det\left(\mathbf{W}^{\mathbf{T}}(\mathbf{x}) d^{\mathbf{2}}_{\mathbf{yy}} H(\mathbf{x}, \mathbf{y})\mathbf{K}(\mathbf{x})\right) \neq \mathbf{0}$$

in some small neighborhood of $\mathbf{x} = \mathbf{0}$ (the matrix $\mathbf{W}^{\mathbf{T}}(\mathbf{0})\mathbf{W}(\mathbf{0})$ is the Grammian matrix of the linearly independent system of vectors $\mathbf{w}^{\mathbf{1}}(\mathbf{0}), \ldots, \mathbf{w}^{\mathbf{d}}(\mathbf{0})$ [66, 83]). Therefore the system of equations (4.21) is solvable with respect to the Lagrange multiplier $\boldsymbol{\lambda}$.

Substituting $\boldsymbol{\lambda}$ into the first group of equations in (4.20), we obtain a system of form (4.1). The force $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ will no longer be linear in $\mathbf{y}$. Note too that, on the invariant manifold defined by the constraint equations, this force is *nonenergetic*, i.e. it follows from (4.19) to (4.20) that

$$\dot{H}(\mathbf{x}, \mathbf{y}) = \left\langle \mathbf{Q}(\mathbf{x}, \mathbf{y}), d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})\right\rangle = \left\langle \boldsymbol{\lambda}, \mathbf{W}^{\mathbf{T}}(\mathbf{x}) d_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})\right\rangle = 0.$$

This property shows that, for the system considered, there must exist energy criteria for stability.

More precisely, we have

Theorem 4.1.7 ([155]). *Let the point $\mathbf{x} = \mathbf{0}$ be a strict local minimum of the potential energy $U(\mathbf{x})$. Then the trivial solution $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ of system (4.1), where the nonpotential force $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ is generated by the constraints (4.19), is Lyapunov stable on the invariant manifold given by the equations of constraint.*

The given theorem is proved using the Lyapunov stability theorem, which here has a conditional character: stability occurs for solutions satisfying initial conditions and subjected to equations of constraint.

Theorem 4.1.7 does not admit a converse in general; it is easy to give an example of a system for which the potential energy doesn't have a minimum at the equilibrium point and thus this equilibrium position is unstable in accordance with the theorem of V.P. Palamodov [148] already cited, but this equilibrium position can become stable upon imposition on the system of a specially chosen form of *holonomic* constraint [40] (and then, for instance, the degree of instability of the original system must equal unity). At the moment it is not known whether an unstable equilibrium position can be stabilized by imposition of a *nonholonomic* constraint. The difficulty lies in the fact that, upon imposition of a holonomic constraint, the dimension of the phase space is actually decreased by two units, whereas imposition of a nonholonomic constraint produces a loss of dimension of only one unit (in fact, if the constraint is described in differential form, and if in the holonomic case the phase space obtained has codimension one, then it can be fibered on the invariant manifolds of codimension two, on only one of which is the equilibrium position in question located). This circumstance leads to the appearance of zero roots for the characteristic equation for the first approximation system. We mention one example that is totally remote from generality, but which shows that even in the nonholonomic case stabilization is possible. This single example will demonstrate that, under imposition of a nonholonomic constraint that yields stability in the first approximation, instability can occur in the nonlinear approximation with a suitable choice of the system parameters.

Example 4.1.2. We consider the motion of a mechanical system with Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \omega^2 x^2 - \omega^2 y^2 + a^2 z^2),$$

the motion being restricted by the nonholomorphic constraint

$$\dot{z} + x\dot{y} = 0.$$

The motion of this system is described by the Vorontsov equations [144]

$$\begin{aligned} &\ddot{x} + \omega^2 x = 0,\\ &\ddot{y} + \frac{x\dot{x}}{1+x^2}\dot{y} + \Omega^2\left(1 - \frac{x^2}{1+x^2}\right)y + \frac{a^2 x}{1+x^2}z = 0,\\ &\dot{z} + x\dot{y} = 0. \end{aligned} \tag{4.22}$$

System (4.22) has two remarkable properties: first, it is linear with respect to $y, \dot{y}, z$. Second, the first equation, describing the change in the variable x, separates and easily integrates. Therefore, after the substitution

$$u = y + \Omega^{-1}\dot{y}, \quad v = y - \Omega^{-1}\dot{y},$$

(4.22) takes the form

$$\begin{aligned}\dot{u} &= -\Omega v + \varepsilon f_3(\varepsilon,t)z + \varepsilon^2\left(f_1(\varepsilon,t)u + f_2(\varepsilon,t)v\right),\\ \dot{v} &= \Omega u - \varepsilon f_3(\varepsilon,t)z - \varepsilon^2\left(f_1(\varepsilon,t)u + f_2(\varepsilon,t)v\right),\\ \dot{z} &= \varepsilon f_4(\varepsilon,t)(u-v).\end{aligned} \tag{4.23}$$

Here the functions f_1, f_2, f_3, f_4 are $\dfrac{2\pi}{\omega}$–periodic in t and can be expanded in power series in ε with nonzero free terms. We introduce the exact formulas:

$$\begin{aligned}f_1(\varepsilon,t) &= -\tfrac{1}{2}\left(\xi(t)\xi(t) + \Omega\xi^2(t)\right)\left(1+\varepsilon^2\xi^2(t)\right)^{-1},\\ f_2(\varepsilon,t) &= -\tfrac{1}{2}\left(\Omega\xi^2(t) - \xi(t)\xi(t)\right)\left(1+\varepsilon^2\xi^2(t)\right)^{-1},\\ f_3(\varepsilon,t) &= -\Omega^{-1}a^2\xi(t)\left(1+\varepsilon^2\xi^2(t)\right)^{-1}, \quad f_4(\varepsilon,t) = -\tfrac{\Omega}{2}\xi(t).\end{aligned}$$

In (4.23) it is taken into account that solutions of the first equation of (4.22), located not far from the equilibrium position, have the form $x(t) = \varepsilon\xi(t)$, where $\xi(t) = \cos(\omega t + \varphi_0)$.

For the sequel we will need the following auxiliary assertion.

Lemma 4.1.1. *Consider a quasiautonomous linear system of differential equations*

$$\dot{\mathbf{x}} = \mathbf{A}(\varepsilon,t)\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^{\mathbf{N}}, \tag{4.24}$$

where the components of the matrix $A(\varepsilon,t)$ $\dfrac{2\pi}{\omega}$ are periodic in t and analytic in ε, and where t is in the interval $(-\varepsilon_0,\varepsilon_0)\times S^1$, $\varepsilon_0 > 0$ sufficiently small.

Let $\lambda_1,\ldots,\lambda_N$ be the eigenvalues of the constant matrix $A_0 \equiv A(0,t)$ and suppose that this matrix is diagonalizable. Further, for all values of the indices $j,l = 1,\ldots,N$ and $p \in Z\setminus\{0\}$, suppose that there are no resonances of the type

$$\lambda_j - \lambda_l = ip\omega. \tag{4.25}$$

Then there exists a formal linear transformation

$$\mathbf{x} = \mathbf{B}(\varepsilon,t)\mathbf{y},$$

where the components of the matrix $\mathbf{B}(\varepsilon,t)$ are expanded in formal power series

$$\mathbf{B}(\varepsilon,t) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty}\mathbf{B}_{\mathbf{k}}(t)\varepsilon^{\mathbf{k}}$$

with $\frac{2\pi}{\omega}$–periodic coefficients, reducing system (4.24) to the autonomous form

$$\dot{\mathbf{y}} = \tilde{\mathbf{A}}(\boldsymbol{\varepsilon})\mathbf{y}, \tag{4.26}$$

where the matrix $A(\varepsilon)$ *is decomposed into a formal series in powers of* ε

$$\tilde{\mathbf{A}}(\varepsilon) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \tilde{\mathbf{A}}_{\mathbf{k}} \varepsilon^k,$$

whereby $\tilde{\mathbf{A}}_\mathbf{0} = \mathbf{A}_\mathbf{0}$.

Remark 4.1.1. In order that the indicated transformation be analytic, it is necessary not just to require the absence of resonances of type (4.25), but also the satisfaction of certain Diophantine equations. However, here we will deal only with the formal aspect of the problem, leaving the question of the convergence of the constructed transformation out of the scope of our investigation.

Proof of Lemma 4.1.1. The matrix $\mathbf{B}(\varepsilon, t)$ must satisfy the following equation:

$$\dot{\mathbf{B}}(\varepsilon, t) = \mathbf{A}(\varepsilon, t)\mathbf{B}(\varepsilon, t) - \mathbf{B}(\varepsilon, t)\tilde{\mathbf{A}}(\varepsilon).$$

In this equation we equate terms with identical powers of ε. For this, we also develop the original matrix $\mathbf{A}(\varepsilon, t)$ in a power series

$$\mathbf{A}(\varepsilon, t) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \mathbf{A}_\mathbf{k}(t)\varepsilon^\mathbf{k}.$$

Clearly we must set $\mathbf{B}_\mathbf{0}(\varepsilon, t) \equiv \mathbf{E}$. Equating terms in ε^k, $k = 1, 2, \ldots$, we obtain a chain of matrix differential equations

$$\dot{\mathbf{B}}_k(t) = \mathbf{L}_{\mathbf{A}_\mathbf{0}}\mathbf{B}_\mathbf{k}(t) + \mathbf{A}_\mathbf{k}(t) - \tilde{\mathbf{A}}_\mathbf{k} + \boldsymbol{\Phi}_\mathbf{k}(t), \quad k = 1, 2, \ldots, \tag{4.27}$$

where

$$\boldsymbol{\Phi}_k(t) = \sum_{s=1}^{k-1} \left(\mathbf{A}_\mathbf{s}(t)\mathbf{B}_{\mathbf{k}-\mathbf{s}}(t) - \mathbf{B}_\mathbf{s}(t)\tilde{\mathbf{A}}_{\mathbf{k}-\mathbf{s}} \right),$$

and we likewise introduce a notation for the commutator matrix

$$\mathbf{L}_{\mathbf{A}_\mathbf{0}}\mathbf{B} = \mathbf{A}_\mathbf{0}\mathbf{B} - \mathbf{B}\mathbf{A}_\mathbf{0}.$$

If $\mathbf{B}_\mathbf{1}(\mathbf{t}), \ldots, \mathbf{B}_{\mathbf{k}-\mathbf{1}}(\mathbf{t})$, $\tilde{\mathbf{A}}_\mathbf{1}, \ldots, \tilde{\mathbf{A}}_{\mathbf{k}-\mathbf{1}}$ have been found, then $\boldsymbol{\Phi}_k(t)$ is a known matrix with $\frac{2\pi}{\omega}$–periodic components.

We consider the Fourier expansions

$$\mathbf{A}_\mathbf{k}(t) = \sum_{\mathbf{p}=-\infty}^{\infty} \mathbf{A}_{\mathbf{kp}} e^{\mathbf{i}\mathbf{p}\omega\mathbf{t}}, \quad \mathbf{B}_\mathbf{k}(\mathbf{t}) = \sum_{\mathbf{p}=-\infty}^{\infty} \mathbf{B}_{\mathbf{kp}} e^{\mathbf{i}\mathbf{p}\omega\mathbf{t}},$$

$$\boldsymbol{\Phi}_\mathbf{k}(t) = \sum_{\mathbf{p}=-\infty}^{\infty} \boldsymbol{\Phi}_{\mathbf{kp}} e^{\mathbf{i}\mathbf{p}\omega\mathbf{t}}.$$

Substituting these expansions into (4.27) and equating terms by respective Fourier harmonics, we obtain

$$i\omega p\mathbf{B_{kp}} = \mathbf{L_{A_0}}\mathbf{B_{kp}} + \mathbf{\Phi_{kp}} + \mathbf{A_{kp}} - \delta_{\mathbf{p0}}\tilde{\mathbf{A}}_{\mathbf{k}}, \tag{4.28}$$

where δ_{p0} is the Kronecker delta.

Lemma 4.1.2. *Consider the N^2–dimensional space of $N \times N$ matrices $\mathbf{B}$ and the operator $\mathbf{L}_{\mathbf{A}_0}$ on this space. The eigenvalues of this matrix are given by $\lambda_j - \lambda_l$, where j, l range independently between 1 and N.*

Proof. Let the matrix $\mathbf{A_0}$ be reduced to diagonal form

$$\mathrm{diag}\,(\lambda_1, \ldots, \lambda_N).$$

Then the matrices $\mathbf{B^{jl}}$ whose j–th row, l–th column entry is one and whose other entries are zero, forms an eigenbasis for the space considered. It is then easily seen that the eigenvalues of the operator $\mathbf{L_{A_0}}$ have the required form.

The lemma is proved.

Since by the hypothesis of Lemma 4.1.1 there are no resonances of type (4.25), by Lemma 4.1.2 the equality (4.28) for $p \neq 0$ is solved in the following way:

$$\mathbf{B_{kp}} = (i\omega p\mathbf{E} - \mathbf{L_{A_0}})^{\mathbf{-1}}(\mathbf{\Phi_{kp}} + \mathbf{A_{kp}}).$$

For $p = 0$ we can obtain

$$\mathbf{B_{k0}} = \mathbf{0}, \quad \tilde{\mathbf{A}}_{\mathbf{k}} = \mathbf{\Phi_{k0}} + \mathbf{A_{k0}}.$$

In this way, the formal transformation that realizes the transition from system (4.24) to the autonomous system (4.26) has been constructed.

Lemma 4.1.1 is proved.

For the system (4.23) considered, the resonances (4.25) will have the form

$$\Omega = \frac{\omega p}{2}, \quad p \in \mathbb{Z} \setminus \{0\}. \tag{4.29}$$

The absence of resonances (4.29) indicates absence in the system of *parametric* resonance. In this case the system of equations (4.23), using the formal linear $\dfrac{2\pi}{\omega}$–periodic substitution $(u, v, z) \mapsto (U, V, Z)$, is reduced to autonomous form. Moreover, without loss of generality we can assume that the resulting autonomous system has been reduced to block form

$$\dot{U} = \alpha(\varepsilon)U - \Omega(\varepsilon)V, \quad \dot{V} = \Omega(\varepsilon)U + \beta(\varepsilon)V, \quad \dot{Z} = \lambda(\varepsilon)Z, \tag{4.30}$$

where

$$\Omega(0) = \Omega, \quad \alpha(0) = \beta(0) = \lambda(0) = 0.$$

In fact, the characteristic equation of the unperturbed system has two pure imaginary roots and one zero, so that the perturbed system theoretically can have two complex conjugate roots (possibly with zero real part) and one real root.

Lemma 4.1.3. *In system (4.30),* $\lambda(\varepsilon) \equiv 0$, $\alpha(\varepsilon) + \beta(\varepsilon) \equiv 0$.

Proof. Since the original mechanical system had the energy integral

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \omega^2 x^2 + \Omega^2 y^2 - a^2 z^2),$$

the system (4.23) has the nonautonomous quadratic integral

$$G = \Omega^2 \left(\left(1 + \varepsilon^2 \xi^2(t)\right)(u - v)^2 + (u + v)^2 \right) - 4a^2 z^2,$$

which, after reduction of the system to autonomous form (4.30), is written

$$\begin{aligned} G = {} & A(\varepsilon, t)U^2 + B(\varepsilon, t)V^2 + C(\varepsilon, t)Z^2 + \\ & + 2D(\varepsilon, t)UV + 2E(\varepsilon, t)UZ + 2F(\varepsilon, t)VZ, \end{aligned}$$

where A, B, C, D, E, F are $\frac{2\pi}{\omega}$–periodic functions, expanded into formal power series in the parameter ε, whereby

$$A(0, t) = B(0, t) = 2\Omega^2 \neq 0, \quad C(0, t) = -4a^2 \neq 0.$$

Since G is the integral of (4.30), the function $C(\varepsilon, t)$ must satisfy the ordinary differential equation

$$\dot{C}(\varepsilon, t) + 2\lambda(\varepsilon)C(\varepsilon, t) = 0$$

and be at least a bounded function of the time.

But this is possible only when $\lambda(\varepsilon) \equiv 0$ and $C(\varepsilon, t) \equiv$const (with respect to time t, for fixed ε). On the other hand, the functions $A(\varepsilon, t)$, $B(\varepsilon, t)$, $D(\varepsilon, t)$ must satisfy the system of three equations

$$\begin{aligned} & \dot{A}(\varepsilon, t) + 2\alpha(\varepsilon)A(\varepsilon, t) + 2\Omega(\varepsilon)D(\varepsilon, t) = 0, \\ & \dot{B}(\varepsilon, t) - 2\Omega(\varepsilon)D(\varepsilon, t) + 2\beta(\varepsilon)B(\varepsilon, t) = 0, \\ & \dot{D}(\varepsilon, t) + (\alpha(\varepsilon) + \beta(\varepsilon))\, D(\varepsilon, t) + \Omega(\varepsilon)\,(B(\varepsilon, t) - A(\varepsilon, t)) = 0, \end{aligned}$$

which possesses a solution bounded in t only if one of the roots of the characteristic equation is zero and the two others are pure imaginary.

The characteristic equation of this system has the form

$$\begin{aligned} \Delta(\rho) = {} & (\rho + (\alpha(\varepsilon) + \beta(\varepsilon))) \times \\ & \times \left(\rho^2 + 2\,(\alpha(\varepsilon) + \beta(\varepsilon))\,\rho + 4\left(\omega^2(\varepsilon) + \alpha(\varepsilon)\beta(\varepsilon)\right)\right) = 0. \end{aligned}$$

The roots of the above characteristic equation satisfy the indicated requirements if and only if $\alpha(\varepsilon) + \beta(\varepsilon) \equiv 0$.

The lemma is proved.

Consequently, for small values of ε, the characteristic equation of system (4.30) will have one zero and two imaginary roots, which indicates stability. Thus the absence of resonances of type (4.29) in the system guarantees stability, at least on the formal level. The presence of a zero root of the characteristic equation of system (4.30) means that, in the five-dimensional phase space of the system, there is a one-dimensional manifold of equilibrium positions, in agreement with the theory presented in [110] on the existence of families of equilibria of the second kind. The trajectories of the system about these equilibria are grouped into invariant tori, whose indestructibility is assured by the incommensurability of the frequencies Ω and ω, and likewise by the presence of a quadratic integral.

In the study of sufficient conditions for stability in a more general setting, it is evident that it is indispensable to use the results recently obtained by M.V. Matveev [138] extending the well-known Arnold-Moser theory on the stability of equilibria of two-dimensional Hamiltonian systems in the general elliptical case [3, 141] for reversible systems.

Thus, by imposition of nonholonomic constraints, it is possible to stabilize unstable equilibria. It should be noted that the hypothesis, stated in the paper [34], that the converse of Theorem 4.1.7 in the particular case of so-called Chaplygin systems [144] is still possible.

After all that has been said it becomes clear that the presence of one of the resonances (4.29) can by itself cause instability. We won't analyze all the resonances of the form (4.29) here and will show merely that instability occurs for the fundamental tone ($p = 2$, $\Omega = \omega$), i.e., for the second frequency of parametric resonance. This case allows us to detect instability already in the *first asymptotic* approximation. The two last equations of (4.22) can be rewritten in the form

$$\begin{aligned} &\ddot{y} + \frac{\varepsilon^2 \xi \xi}{1+\varepsilon^2\xi^2}\dot{y} + \omega^2\left(1 - \frac{\varepsilon^2\xi^2}{1+\varepsilon^2\xi^2}\right)y + \frac{\varepsilon a^2 \xi}{1+\varepsilon^2\xi^2}z = 0, \\ &\dot{z} + \varepsilon\xi\dot{y} = 0, \end{aligned} \tag{4.31}$$

where $\xi = \cos\phi$, and $\phi = \omega t$ is the phase of parametric excitation.

We use an algorithm close to what is described in a classical work on perturbation theory [18]. The solutions of system (4.31) can be sought in the form of power series in a small parameter ε:

$$y = I\cos\psi + \sum_{k=1}^{\infty} Y_k(I, J, \phi, \theta)\varepsilon^k, \quad z = J + \sum_{k=1}^{\infty} Z_k(I, J, \phi, \theta)\varepsilon^k.$$

Here I, ψ are amplitude and phase of nonlinear oscillations, and $\theta = \psi - \phi$ is the phase shift. We choose functions $Y_k, Z_k, k = 1, 2, \ldots$, periodic with period 2π

in the last two variables ϕ, ψ, in such a way that the quantities I, J, θ satisfy a system of differential equations of the form:

$$\dot{I} = \sum_{k=1}^{\infty} A_k(I, J, \theta)\varepsilon^k, \ \dot{J} = \sum_{k=1}^{\infty} B_k(I, J, \theta)\varepsilon^k, \ \dot{\theta} = \sum_{k=1}^{\infty} C_k(I, J, \theta)\varepsilon^k,$$

where the functions $A_k, B_k, C_k, k = 1, 2, \ldots$, depend on θ 2π–periodically.

Leaving out details, we mention formulae that are part of the construction of the first approximation in ε:

$$A_1 = \frac{a^2 J \sin\theta}{2\omega}, \quad B_1 = \frac{\omega I \sin\theta}{2}, \quad C_1 = \frac{a^2 J \cos\theta}{2\omega I},$$
$$Y_1 = 0, \quad Z_1 = -\frac{l\cos(\theta + 2\phi)}{4}.$$

Passing finally from "polar" coordinates I, ψ to "Cartesian" η, ς

$$\eta = I\cos\theta, \quad \varsigma = I\sin\theta,$$

we obtain that the original system (4.31) takes the following form:

$$\dot{\eta} = O(\varepsilon^2), \quad \dot{\varsigma} = \frac{\varepsilon a^2}{2\omega} J + O(\varepsilon^2), \quad \dot{J} = \frac{\varepsilon\omega}{2}\varsigma + O(\varepsilon^2), \tag{4.32}$$

where we have written out only the first order terms in ε.

The "truncated system"—for which we have rejected all terms on the right sides with order in ε greater than one—is a linear system of differential equations with constant coefficients. The characteristic equation for this system has one zero root (which corresponds to a one-dimensional equilibrium manifold mentioned earlier) and two real nonzero roots of opposite sign: $\pm\frac{\varepsilon a}{2}$. The presence of these roots results in weak (order ε) exponential instability of the equilibrium point of system (4.32) and consequently leads to instability for the original system.

Returning now to consideration of the general problem, a nontrivial situation arises when, at the equilibrium position $\mathbf{x} = \mathbf{0}$, there is a loss of rank for the matrix $\mathbf{W}(\mathbf{x})$. In this case system (4.19) either can't in general be reduced to the form (4.10) or the nonpotential force $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ will have a singularity. Moreover, at the equilibrium position (which is of the first kind!), the Lagrange multiplier λ can depend on some arbitrary parameter. For such a system the Lyapunov stability problem doesn't make any general sense, since the hypothesis of the theorem on existence and uniqueness of solutions in the neighborhood of this equilibrium position is violated. Nonetheless, in the next section we will indicate an additional condition for *instability* which will guarantee that solutions starting in a sufficiently small neighborhood of the equilibrium position will exit some fixed neighborhood of this equilibrium position.

4.2 Regular Problems

This section is dedicated to an analysis of those problems of treating energy criteria for stability, formulated in the preceding sections, which lead to an analysis of systems that are positive semi-quasihomogeneous by the "classical" Definition 1.1.4.

We begin by studying the generalized-gradient systems (4.2). Let

$$U(\mathbf{x}) = \sum_{\mathbf{m}=\mathbf{M}} U_m(\mathbf{x}), \quad M \geq 2,$$

be an expansion of the potential into a Maclaurin series in the neighborhood of $\mathbf{x} = \mathbf{0}$.

In this section we concentrate on situations where the expansion of $U(\mathbf{x})$ begins with a form of at least *third* degree.

We consider first the classical *gradient* case:

$$\dot{\mathbf{x}} = -\mathbf{D}^{-1}(\mathbf{x})\, dU(\mathbf{x}), \tag{4.33}$$

where $\mathbf{D}(\mathbf{x})$ is a symmetric matrix that is positive definite for all values of $\mathbf{x}$ in some neighborhood of the origin.

We have

Theorem 4.2.1 ([106]). *Suppose the first nontrivial form $U_M(\mathbf{x})$ of the expansion of the potential energy in a Maclaurin series can take on negative values. Then the critical point of the system considered is unstable: there is a particular solution of (4.33) $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$ for which there exists a finite limit*

$$\lim_{t \to -\infty} \frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|} = \mathbf{e}, \tag{4.34}$$

where $\mathbf{e} \in \mathbb{R}^n$ is some unit vector.

It should be noted that the existence of an asymptotic trajectory as $t \to -\infty$, under the condition that the potential energy doesn't have a minimum at the *isolated* critical point considered, follows from the theorem of N.N. Krasovskiy [124] that was mentioned in the introduction to this book. But, more interesting here, is the assertion of the existence of a limit (4.34). The direction given by the vector $\mathbf{e}$ is called characteristic. RenŽ Thom [67] made the suggestion that if a potential gradient system didn't have a local minimum at an isolated critical point, then there exists an asymptotic solution with characteristic direction. This conjecture has not yet been proven in all generality.

Proof of Theorem 4.2.1. The coordinate system in which (4.33) is written can be chosen so that $\mathbf{D(0)} = \mathbf{E}$. We consider the standard scale of homogeneous dilations on $\mathbb{R}^n$ ($\mathbf{G} = \frac{1}{M-2}\mathbf{E}$). The corresponding truncated system (4.33) will have the form

$$\dot{\mathbf{x}} = -dU_M(\mathbf{x}). \tag{4.35}$$

Under the hypothesis of the theorem, system (4.35) has a particular solution in the form of an increasing linear ray:

$$\mathbf{x}^- = (-t)^{-\alpha}\mathbf{x}_0^-, \quad \alpha = \frac{1}{M-2}, \quad \mathbf{x}_0^- \in \mathbb{R}^\mathbf{n}.$$

Let $\mathbf{e} \in \mathbb{R}^\mathbf{n}$, $\|\mathbf{e}\| = 1$ be the vector at which the function $U_M(\mathbf{x})$ takes its minimum value on the sphere

$$S^{n-1} = \{\mathbf{p} \in \mathbb{R}^\mathbf{n}: \|\mathbf{p}\| = 1\}.$$

We consider the Lagrange function

$$\Phi(\mathbf{x},a) = \mathbf{U_M}(\mathbf{x},a) + \frac{a}{2}(\|\mathbf{x}\|^2 - 1). \tag{4.36}$$

Since, at a relative extremum, $d\Phi(\mathbf{e},a) = \mathbf{0}$ we have

$$dU_M(\mathbf{e}) = -a\mathbf{e}. \tag{4.37}$$

Taking the scalar product of (4.37) with $\mathbf{e}$ and applying Euler's theorem on homogeneous functions, we obtain

$$a = -M \min_{\mathbf{p}\in S^{\mathbf{n-1}}} U_M(\mathbf{p}) > 0.$$

Setting $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$ and using (4.37), we find that

$$\|\mathbf{x}_0^-\| = \left(\frac{\alpha}{a}\right)^\alpha.$$

Thus the hypothesis of Theorem 1.1.2 is satisfied, from which it follows that the system of equations (4.33) has a particular solution with asymptotic expansion

$$\mathbf{x}(t) = \sum_{k=0}^{\infty} \mathbf{x}_k\,(\ln(-t))\,(-t)^{-\alpha(k+1)}, \quad \mathbf{x_0} = \mathbf{x}_0^-. \tag{4.38}$$

The theorem is proved.

Remark 4.2.1. If the potential $U(\mathbf{x})$ and the components of the matrix $\mathbf{D}(\mathbf{x})$ are analytic, then series (4.38) converges for $t \in (-\infty, -T]$, where $T > 0$ is sufficiently large.

For the proof we apply Theorem 1.2.1. We estimate the eigenvalues of the Kovalevsky matrix:

$$\mathbf{K} = \alpha\mathbf{E} + d^2\mathbf{U_M}(\mathbf{x}_0^-) = \alpha\left(\mathbf{E} + a^{-1}\,d^2U_M(\mathbf{e})\right).$$

For this we use the properties of the second variation of the Lagrange function (4.36): for any $\boldsymbol{\xi} \in T_{\mathbf{e}}S^{n-1}$,

$$\left\langle d^2_{\mathbf{xx}}\Phi(\mathbf{e},\mathbf{a})\boldsymbol{\xi},\boldsymbol{\xi}\right\rangle = \frac{a}{\alpha}\langle \mathbf{K}\boldsymbol{\xi},\boldsymbol{\xi}\rangle \geq 0.$$

Therefore all eigenvalues of the Kovalevsky matrix (which is symmetric in this instance) are positive, with the single exception of the eigenvalue -1.

This eigenvalue is the only one that satisfies a resonance relation of the form $\rho = -k\alpha$, $k \in \mathbb{N}$, so that all the hypothesis of Theorem 1.2.1 is satisfied, which is what we needed to prove.

For a generalized–gradient system an assertion analogous to Theorem 4.2.1 has not yet been established. Moreover, for systems having the Hamiltonian form (4.5), this assertion is simply false. We have the weaker result:

Theorem 4.2.2. *Let the degree of the first nontrivial form $U_M(\mathbf{x})$ of the expansion of potential energy in a Maclaurin series be odd and let the critical point $\mathbf{x} = \mathbf{0}$ of the form $U_M(\mathbf{x})$ be isolated. Then the critical point of the generalized–gradient system (4.2) is unstable and there exists a particular asymptotic solution of (4.2), $\mathbf{x(t)} \to \mathbf{0}$ as $t \to -\infty$, for which the limit (4.34) exists and is finite.*

Proof. As in the previous case we separate a truncation from (4.2) by using the group of homogeneous dilations generated by the matrix $\mathbf{G} = \frac{1}{M-2}\mathbf{E}$. This truncation will have the form

$$\dot{\mathbf{x}} = \mathbf{B^{-1}}\, dU_{\mathbf{M}}(\mathbf{x}), \tag{4.39}$$

where as before we use the notation $\mathbf{B} = \mathbf{B(0)}$.

The fact that the truncated system (4.39) has an increasing linear solution $\mathbf{x}^-(t) = (-t)^{-\boldsymbol{\alpha}}\,\mathbf{x_0^-}$, $\alpha = \frac{1}{M-2}$ in the form of a ray follows from Lemma 1.1.1. In fact, since the degree $M-1$ of the homogeneous vector field $\mathbf{B^{-1}}\,dU_M(\mathbf{x})$ is even, we have that the index of this vector field is even and that it has eigenvectors with both positive and negative eigenvalues. To complete the proof we apply Theorem 1.1.2.

The theorem is proved.

Critical instability as formulated in Theorem 4.2.2 can also be related to energy, since in this case the potential energy in the neighborhood of the critical point considered is of variable sign. Unfortunately the eigenvalues of the Kovalevsky matrix in the given case can't be estimated as in the proof of Theorem 4.2.1, so that we can't draw a conclusion about the convergence of the series (4.38) that represents the original solution.

We proceed to find instability criteria for equilibrium positions of mechanical systems, whose motion is described by systems of equations of type (4.1). Just as for the previous problem, we assume that the expansion of potential energy in a

Maclaurin series begins with terms of at least third order. We expand the matrix $\mathbf{B}(\mathbf{x})$ in a Maclaurin series in the neighborhood of the equilibrium position:

$$\mathbf{B}(\mathbf{x}) = \sum_{\mathbf{s}=\mathbf{S}}^{\infty} \mathbf{B_s}(\mathbf{x}), \quad S \geq 0,$$

where the components of the matrix $\mathbf{B_s}(\mathbf{x})$ are homogeneous functions of $\mathbf{x}$ of degree s.

We will study the action on the system of a strongly singular nonpotential force, i.e. we will assume that $S > 0$.

Theorem 4.2.3. *Suppose the first nontrivial form $U_M(\mathbf{x})$, $M \geq 3$, in the expansion of the potential energy into a Maclaurin series doesn't have a minimum, and that $S > [(M-2)/2]$. Then the equilibrium solution $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ of system (4.1) is unstable and there exists a particular asymptotic solution of (4.1): $\mathbf{x}(t) \to \mathbf{0}$, $\mathbf{y}(t) \to \mathbf{0}$ as $t \to -\infty$.*

In the absence of nonpotential forces this result was proved in the papers [104, 117] and, for the case of singular gyroscopic forces, in [53, 54].

Proof. The system of equations (4.1) is positive semi-quasihomogeneous with respect to the structure given by the diagonal matrix

$$\mathbf{G} = \mathrm{diag}\,(\underbrace{M\alpha, \dots, M\alpha}_{y}, \underbrace{2\alpha, \dots, 2\alpha}_{x}), \quad \alpha = \frac{1}{M-2}, \quad \beta = \alpha.$$

Since the system considered is written in normal coordinates, i.e. $\mathbf{K}(\mathbf{0}) = \mathbf{E}$, and since we have the inequality $S > [(M-2)/2]$, it is not difficult to see that the corresponding quasihomogeneous truncation has the form

$$\dot{\mathbf{y}} = -dU_M(\mathbf{x}), \quad \dot{\mathbf{x}} = \mathbf{y}. \tag{4.40}$$

The system of equations (4.40) is quasihomogeneous in the sense of Definition 1.1.2, with indices $M, \dots, M, 2, \dots, 2$ and degree $q = M - 1$. If the hypothesis of Theorem 4.2.3 is satisfied, then this system has an increasing solution in the form of a quasihomogeneous ray:

$$\mathbf{x}^-(t) = (-t)^{-2\alpha}\mathbf{x}_0^-, \quad \mathbf{y}^-(t) = (-t)^{-\mathbf{M}\alpha}\mathbf{y}_0^-,$$
$$\mathbf{y}_0^- = 2\alpha\mathbf{x}_0^-, \quad \mathbf{x}_0^- \in \mathbb{R}^{\mathbf{n}}.$$

We set $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^{\mathbf{n}}$, $\|\mathbf{e}\| = \mathbf{1}$, is a vector for which the function $U_M(\mathbf{x})$ assumes a minimum value on the sphere $S^{n-1} = \{\mathbf{p} \in \mathbb{R}^{\mathbf{n}} \colon \|\mathbf{p}\| = \mathbf{1}\}$. Using equality (4.37), we obtain

$$\mathbf{x}_0^- = \left(\frac{2M\alpha^2}{a}\right)^{\alpha}\mathbf{e}.$$

Thus, all the hypothesis of Theorem 1.1.2 is satisfied and it follows that the system of equations (4.1) has a particular solution with the asymptotic expansions

$$\begin{aligned} \mathbf{x}(t) &= \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \mathbf{x}_{\mathbf{k}}\,(\ln(-t))\,(-t)^{-\alpha(\mathbf{k}+\mathbf{2})}, \quad \mathbf{x}_{\mathbf{0}} = \mathbf{x}_{\mathbf{0}}^{-}, \\ \mathbf{y}(t) &= \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \mathbf{y}_{\mathbf{k}}\,(\ln(-t))\,(-t)^{-\alpha(\mathbf{k}+\mathbf{M})}, \quad \mathbf{y}_{\mathbf{0}} = \mathbf{y}_{\mathbf{0}}^{-}. \end{aligned} \tag{4.41}$$

The theorem is proved.

We note that if the system is analytic, i.e. if we have analyticity for potential energy $U(\mathbf{x})$ and also for the components of the matrices $\mathbf{K}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$, then the series in (4.41) converge for $t \in (-\infty, -T]$, where $T > 0$ is sufficiently large. In fact, in the case considered, the Kovalevsky matrix has the following block form:

$$\mathbf{K} = \begin{pmatrix} M\alpha\mathbf{E} & d^2U_M(\mathbf{x}_0^-) \\ -\mathbf{E} & 2\alpha\mathbf{E} \end{pmatrix}$$

(which should not be confused with the kinetic energy matrix!).

Performing elementary transformations on the rows of the corresponding determinant, we can show that the eigenvalues of the matrix $\mathbf{K}$ satisfies the equation

$$\det\left(d^2U_M(\mathbf{x}_{\mathbf{0}}^-) + (\rho - 2\alpha)(\rho - M\alpha)\mathbf{E}\right) = 0.$$

Let $k_1, \dots, k_n$ be the eigenvalues of the symmetric matrix

$$\mathbf{K}^* = 2M\alpha^2\mathbf{E} + d^2U_M(\mathbf{x}_{\mathbf{0}}^-).$$

On the one hand, using the Euler homogeneous function theorem and equality (4.37), we obtain

$$\mathbf{K}^*\mathbf{e} = \|\mathbf{x}_{\mathbf{0}}^-\|^{\mathbf{M}-\mathbf{2}}(M-1)dU_M(\mathbf{e}) + 2M\alpha^2\mathbf{e} = -2M\alpha\mathbf{e},$$

i.e. we may assume that $k_1 = -2M\alpha$.

On the other hand, we can again use the property of the second variation of the Lagrange function (4.36): for any $\boldsymbol{\xi} \in T_{\mathbf{e}}S^{n-1}$

$$\left\langle d^2_{\mathbf{xx}}\Phi(\mathbf{e}, \mathbf{a})\boldsymbol{\xi}, \boldsymbol{\xi}\right\rangle = \frac{a}{2M\alpha^2}\langle \mathbf{K}^*\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \geq 0,$$

i.e. $k_2, \dots, k_n$ are nonnegative.

Consequently the first two eigenvalues of the Kovalevsky matrix are given by

$$\rho_1 = 1, \quad \rho_{n+1} = 2M\alpha,$$

and the remaining are

$$\rho_{j,n+j} = \frac{M+2}{2(M-2)} \left(1 \pm \sqrt{1 - 4k_j \left(\frac{M-2}{M+2} \right)^2} \right), \quad j = 2, \ldots, n.$$

If the quantity under the radical is nonnegative, then the numbers $\rho_{j,n+j}$ will be positive, and if it is negative the eigenvalues will be complex. In any case $\rho_1 = -1$ is the only eigenvalue of the Kovalevsky matrix of the form $-k\alpha$, so that the hypothesis of Theorem 1.2.1 is satisfied in its entirety, which was to be shown.

Theorem 4.2.3 is of course also true for the case $M = 2$ $(S \geq 1)$. This is easily derived from the Lyapunov instability theorem with respect to the first approximation [133]. For the case of gyroscopic forces, the corresponding assertion was formulated in the paper by L. Salvadori [157].

Theorem 4.2.3 proves the assertion, known in physics as Earnshaw's theorem [183], that any equilibrium of a charge in an electrostatic field is always unstable. In fact, let $\mathbf{x} = \mathbf{0}$, $\mathbf{x} \in \mathbb{R}^3$ be the equilibrium position of the charge considered. We expand the potential energy of the electrostatic field in the neighborhood of this equilibrium:

$$U(\mathbf{x}) = \sum_{\mathbf{m}=\mathbf{M}} U_{\mathbf{M}}(\mathbf{x}).$$

From electrostatics we know that the function $U(\mathbf{x})$ is harmonic:

$$(\Delta U)(\mathbf{x}) \equiv \mathbf{0},$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is the Laplace operator. Therefore all the forms of its Maclaurin expansion are harmonic, in particular the form $U_M(\mathbf{x})$. According to the maximum principle for harmonic functions (see e.g. [129]), $U_M(\mathbf{x})$ is of variable sign. Then for $M = 2$ the instability result follows from the classical Lyapunov theorem [131], and for $M > 2$ from Theorem 4.2.3. This interesting consequence was first formulated in the paper [106].

In the paper [108] Earnshaw's theorem was considered for the case where the force function V (the negative of the potential energy) is *subharmonic*: $\Delta V \geq 0$. More precisely, let $\mathbf{x} = \mathbf{0}$—an equilibrium position—be a critical point of the smooth subharmonic force function and let its Maclaurin series be nonzero at the point $\mathbf{x} = \mathbf{0}$. Then this equilibrium position is unstable. In the analytic case the subharmonic condition is sufficient for instability.

We now consider the problem of finding sufficient conditions for the instability of an equilibrium position of the first kind for a mechanical system, whose motion is restricted by some nonholonomic constraints. We recall that the *stability*

Theorem 4.1.7 does not admit a full converse. The result below was formulated and proved in [105].

Theorem 4.2.4. *Consider the* $(n-d)$*–dimensional subspace*

$$\pi = \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}}: \mathbf{W}^{\mathbf{T}}(\mathbf{0})\mathbf{x} = \mathbf{0}\} \tag{4.42}$$

and the homogeneous form $V_M(\mathbf{x})$ *on the subspace* π*, which is the restriction of the form* $U_M(\mathbf{x})$*,* $M \geq 3$*, to the subspace* π $(V_M = U_M|_{\mathbf{x}\in\pi})$*. If the form* $V_M(\mathbf{x})$ *does not have a minimum, then the equilibrium solution of system (4.1)* $\mathbf{x} = \mathbf{0}$*,* $\mathbf{y} = \mathbf{0}$*, where the nonpotential force* $Q(\mathbf{x},\mathbf{y})$ *is generated by the constraints (4.19), is unstable and there exists a particular asymptotic solution of (4.1),* $\mathbf{x(t)} \to \mathbf{0}$*,* $\mathbf{y(t)} \to \mathbf{0}$ *as* $t \to -\infty$*, which lies on the invariant manifold given by the equations of constraint (4.19).*

Proof. As in the preceding case, it is easily observed that the system of equations (4.1) is positive semi-quasihomogeneous with respect to the structure given by the diagonal matrix

$$\mathbf{G} = \operatorname{diag}(\underbrace{M\alpha,\ldots,M\alpha}_{y},\underbrace{2\alpha,\ldots,2\alpha}_{x}), \quad \alpha = \frac{1}{M-2}, \quad \beta = \alpha.$$

The corresponding truncation has the form

$$\dot{y} = -dU_M(\mathbf{x}) + \mathbf{W}(\mathbf{0})\left(\mathbf{W}^{\mathbf{T}}(\mathbf{0})\mathbf{W}(\mathbf{0})\right)^{-\mathbf{1}}\mathbf{W}^{\mathbf{T}}(\mathbf{0})dU_M(\mathbf{x}), \tag{4.43}$$

$$\dot{\mathbf{x}} = \mathbf{y}. \tag{4.44}$$

We prove that, if the hypothesis of Theorem 4.2.4 is satisfied, then the truncated system (4.43) has an increasing solution in the form of a quasihomogeneous ray

$$\mathbf{x}^-(t) = (-t)^{-2\alpha}\mathbf{x}_0^-, \quad \mathbf{y}^-(t) = (-t)^{-\mathbf{M}\boldsymbol{\alpha}}\mathbf{y}_0^-,$$
$$\mathbf{y}_0^- = \mathbf{2}\boldsymbol{\alpha}\mathbf{x}_0^-, \quad \mathbf{x}_0^- \in \mathbb{R}^{\mathbf{n}}.$$

Since the form $V_M(\mathbf{x})$, $\mathbf{x} \in \boldsymbol{\pi}$ doesn't have a minimum, we can set $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^{\mathbf{n}}$ and $\|\mathbf{e}\| = \mathbf{1}$ is a vector for which the function $U_M(\mathbf{x})$ takes on a minimum value on the set $\pi \cap S^{n-1} = \{\mathbf{p} \in \mathbb{R}^{\mathbf{n}}: \|\mathbf{p}\| = \mathbf{1}, \mathbf{W}(\mathbf{0})\mathbf{p} = \mathbf{0}\}$.

Since the equation that gives the subspace π has the form (4.42), the corresponding Lagrange function equals

$$\Phi(\mathbf{x},a,\boldsymbol{\mu}) = U_M(\mathbf{x}) + \frac{a}{2}\left(\|\mathbf{x}\|^{\mathbf{2}} - \mathbf{1}\right) - \langle\mathbf{W}(\mathbf{0})\boldsymbol{\mu},\mathbf{x}\rangle, \tag{4.45}$$

where $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_n)$ are Lagrange multipliers. At a critical extreme point $d\Phi(\mathbf{x},a,\boldsymbol{\mu}) = \mathbf{0}$, so that

$$dU_M(\mathbf{e}) = -a\mathbf{e} + \mathbf{W}(\mathbf{0})\boldsymbol{\mu}. \tag{4.46}$$

We take the scalar product of (4.46) with $\mathbf{e}$ and apply Euler's homogeneous function theorem to obtain

$$a = -M \min_{\mathbf{p}\in\mathbf{S^{n-1}}\cap\pi} U_M(\mathbf{p}) > \mathbf{0}.$$

From (4.46) we find the value of $\boldsymbol{\mu}$, and we apply the operator with matrix $\mathbf{W^T}(\mathbf{0})$ to the left and right sides of (4.46). Since $\mathbf{e} \in \boldsymbol{\pi}$, we have $\mathbf{W^T}(\mathbf{0})\mathbf{e} = \mathbf{0}$. Consequently,

$$\boldsymbol{\mu} = \left(\mathbf{W^T}(\mathbf{0})\mathbf{W}(\mathbf{0})\right)^{-1}\mathbf{W^T}(\mathbf{0})\, dU_M(\mathbf{e}).$$

We can now compute the value of $\|\mathbf{x_0^-}\|$ by the formula

$$\|\mathbf{x_0^-}\| = \left(\frac{2M\alpha^2}{a}\right)^{\alpha},$$

so that all the hypothesis of Theorem 1.1.2 is satisfied and it follows that Eq. (4.1) has a particular solution with asymptotic expansion (4.41). To complete the proof we need to show that the determined asymptotic trajectory $(\mathbf{x(t)}, \mathbf{y(t)})$ lies on the invariant manifold given by the Eq. (4.52). It is clear that as $t \to -\infty$ we have

$$\mathbf{g}(t) = \mathbf{W^T}\,(\mathbf{x(t)})\, d_{\mathbf{y}}H\,(\mathbf{x(t)}, \mathbf{y(t)}) \to \mathbf{0}.$$

We compute the time derivative of $\mathbf{g}(t)$:

$$\begin{aligned}\dot{\mathbf{g}}(t) &= \boldsymbol{\Gamma}\left(\mathbf{x}(t), d_{\mathbf{y}}H\,(\mathbf{x}(t), \mathbf{y}(t))\right) + \\ &+\mathbf{W^T}\,(\mathbf{x}(t))\left[d^2_{\mathbf{yy}}H\,(\mathbf{x}(t), \mathbf{y}(t))\,(-d_{\mathbf{x}}H\,(\mathbf{x}(t), \mathbf{y}(t))) + \mathbf{Q}H\,(\mathbf{x}(t), \mathbf{y}(t)) + \right.\\ &\left. +\mathbf{d^2_{yy}}H\,(\mathbf{x}(t), \mathbf{y}(t))\, d_{\mathbf{y}}H\,(\mathbf{x}(t), \mathbf{y}(t))\right]\end{aligned}$$

Taking into account, in accordance with the construction of $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ (see e.g. (4.43)), that

$$\boldsymbol{\Gamma} + \mathbf{W^T}\left(d^2_{\mathbf{yy}}H(-d_{\mathbf{x}}H + \mathbf{Q}) + d^2_{\mathbf{xy}}H d_{\mathbf{y}}H\right) \equiv \mathbf{0},$$

we have $\dot{\mathbf{g}}(t) \equiv \mathbf{0}$ and hence $\mathbf{g(t)} \equiv \mathbf{0}$.

The theorem is proved.

The theorem we have proved is of course also true for $M = 2$, for which the corresponding assertion is due to Whittaker [193].

To conclude, we will continue with the case where the matrix $\mathbf{W}(\mathbf{0})$ has rank less than d. We consider the simplest case where there is but one constraint imposed on the system:

$$\left\langle \mathbf{w}(\mathbf{x}), \mathbf{d_y H}(\mathbf{x}, \mathbf{y})\right\rangle = 0, \quad \mathbf{w}(\mathbf{0}) = \mathbf{0} \tag{4.47}$$

In this case the vector of Lagrange multipliers is one-dimensional. The system of equations (4.20) can also be solved with respect to the multiplier λ, but the system obtained will have a singularity. In fact, differentiating (4.47) with respect to time and substituting the expressions for $\dot{\mathbf{x}}, \dot{\mathbf{y}}$ from system (4.20), we obtain

$$\lambda = \left(\left\langle d^2_{\mathbf{yy}} H(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x}), \mathbf{w}(\mathbf{x})\right\rangle\right)^{-1} \left(d^2_{\mathbf{yy}}\mathbf{H}(\mathbf{x},\mathbf{y}) d_{\mathbf{x}}\mathbf{H}(\mathbf{x},\mathbf{y}) - \right. \\ \left. - d^2_{\mathbf{yx}} H(\mathbf{x},\mathbf{y}) d_{\mathbf{y}}\mathbf{H}(\mathbf{x},\mathbf{y}), \mathbf{w}(\mathbf{x})\rangle - \left\langle d\mathbf{w}(\mathbf{x}) d_{\mathbf{y}}\mathbf{H}(\mathbf{x},\mathbf{y}), d_{\mathbf{y}}\mathbf{H}(\mathbf{x},\mathbf{y})\right\rangle\right)$$

We let σ define the quantity

$$\sigma = (\langle \mathbf{K}(\mathbf{x})\mathbf{w}(\mathbf{x}), \mathbf{w}(\mathbf{x})\rangle)^{-1,}$$

and consider the extended system of equations

$$\begin{aligned} \dot{\mathbf{y}} &= -d_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) + \text{æ}(\Big\langle d^2_{yy} H(\mathbf{x},\mathbf{y}) d_x H(\mathbf{x},\mathbf{y}) - \\ &\quad - d^2_{yx} H(\mathbf{x},\mathbf{y}) d_y H(\mathbf{x},\mathbf{y}), \mathbf{w}(\mathbf{x})\Big\rangle \\ &\quad - \left\langle d\mathbf{w}(\mathbf{x}) d_y H(\mathbf{x},\mathbf{y}), d_y H(\mathbf{x},\mathbf{y})\right\rangle)\mathbf{w}(\mathbf{x}), \\ \dot{\mathbf{x}} &= d_y H(\mathbf{x},\mathbf{y}), \\ \dot{\sigma} &= -\sigma^2 \left\langle \mathbf{\Theta}\,(\mathbf{x},\mathbf{y},\mathbf{w}(\mathbf{x})) + 2\mathbf{K}(\mathbf{x})\mathbf{d}\mathbf{w}(\mathbf{x})\mathbf{K}(\mathbf{x})\mathbf{y}, \mathbf{w}(\mathbf{x})\right\rangle, \end{aligned} \tag{4.48}$$

where $\mathbf{\Theta}(\mathbf{x},\mathbf{y},\cdot)$ is a certain mapping, quadratic in its third argument, whose coefficients depend linearly on $\mathbf{y}$, and where the coefficients of the corresponding linear forms are the partial derivatives of the components of the matrix $\mathbf{K}(\mathbf{x})$.

System (4.48) no longer has a singularity, but has the point $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, $\sigma = 0$ is a critical point (equilibrium position).

We consider the expansion of the vector field $\mathbf{w}(\mathbf{x})$ in a power series in the neighborhood of the origin:

$$\mathbf{w}(\mathbf{x}) = \sum_{r=R}^{\infty} \mathbf{w}_r(\mathbf{x}), \quad R \geq 1.$$

Theorem 4.2.5 ([63]). *Suppose the following conditions are satisfied:*

1. *The constraint (4.47) imposed on the system is weakly nonholonomic* (terminology due to Ya.V. Tamarinov)*, i.e. the first nontrivial form in the expansion of the vector field $w(x)$ is the gradient of some homogeneous function:*

$$\mathbf{w}_R(\mathbf{x}) = d\varphi_{R+1}(\mathbf{x});$$

2. *The set*

$$\pi = \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \varphi_{R+1}(\mathbf{x}) = 0,\ \mathbf{x} \neq \mathbf{0}\} \tag{4.49}$$

is nonempty and the point $\mathbf{x} = \mathbf{0}$ is the unique critical point of the homogeneous function $\varphi_{R+1}(\mathbf{x})$;

3. *The homogeneous function $V_M(x)$, defined on the set π, which is the restriction of the form $U_M(\mathbf{x})$, $M \geq 3$, to the set π ($V_M = U_M|_{\mathbf{x}\in\pi}$), doesn't have a minimum.*

Then there exists a particular solution of system (4.1) (the nonpotential force $Q(x, y)$ is determined by the constraint (4.47)) that lies on the invariant manifold generated by the constraint and is such that $\mathbf{x(t)} \to \mathbf{0}$, $\mathbf{y(t)} \to \mathbf{0}$ *as* $t \to -\infty$.

Proof. The system of equations (4.48) is a consequence of system (4.1) and can be regarded as semi-quasihomogeneous with respect to the structure generated by the diagonal matrix

$$\mathbf{G} = \operatorname{diag}(\underbrace{M\alpha, \dots, M\alpha}_{y}, \underbrace{2\alpha, \dots, 2\alpha}_{x}, \underbrace{-4R\alpha}_{\sigma}),$$
$$\alpha = \tfrac{1}{M-2}, \quad \beta = \alpha.$$

We write its truncation:

$$\begin{aligned} \dot{\mathbf{y}} &= -\,dU_M(\mathbf{x}) + \sigma\left(\langle dU_M(\mathbf{x}), \mathbf{w}_R(\mathbf{x})\rangle \langle \mathbf{dw}_R(\mathbf{x})\mathbf{y}, \mathbf{y}\rangle\right)\mathbf{w}_R(\mathbf{x}), \\ \dot{\mathbf{x}} &= \mathbf{y}, \\ \dot{\sigma} &= -2\sigma^2 \langle d\,\mathbf{w}_R(\mathbf{x})y, \mathbf{w}_R(\mathbf{x})\rangle . \end{aligned} \tag{4.50}$$

The system of equations (4.50) is quasihomogeneous by Definition 1.1.2, with indices $M, \dots, M, 2, \dots, 2, -4R$ and exponent $q = M - 1$. We show that, when the hypothesis of Theorem 4.2.5 is fulfilled, the system of equations (4.50) has a particular solution in the form of a quasihomogeneous ray:

$$\mathbf{x}^-(t) = (-t)^{-2\alpha}\mathbf{x}_0^-, \;\; \mathbf{y}^-(t) = (-t)^{-M\alpha}\mathbf{y}_0^-, \;\; \sigma^-(t) = (-t)^{4R\alpha}\sigma_0^-,$$
$$\mathbf{y}_0^- = 2\alpha\mathbf{x}_0^-, \quad \mathbf{x}_0^- \in \mathbb{R}^n, \quad \sigma_0^- \in \mathbb{R}.$$

We note that, in system (4.50), $\mathbf{w}_R(\mathbf{x})$, $d\,\mathbf{w}_R(\mathbf{x})$ can be replaced by $d\varphi_{R+1}(\mathbf{x})$, $d^2\varphi_{R+1}(\mathbf{x})$. We set $\mathbf{x_0^-} = \|\mathbf{x_0^-}\|\mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^\mathbf{n}$, $\|\mathbf{e}\| = \mathbf{1}$, is the vector at which the function $U_M(\mathbf{x})$ assumes its minimum value on the set $\pi \cap S^{n-1} = \{\mathbf{p} \in \mathbb{R}^\mathbf{n}\colon \|\mathbf{p}\| = \mathbf{1}, \varphi_{R+1}(\mathbf{p}) = \mathbf{0}\}$, and

$$\sigma_0^- = \|d\varphi_{R+1}(\mathbf{x_0^-})\|^{-\mathbf{2}}.$$

From the first equation of the truncated system and Euler's theorem on homogeneous functions, we have the equality

$$\begin{aligned} &dU_M(\mathbf{x_0^-}) + 2M\alpha^2\mathbf{x_0^-} = \\ &\quad = \|d\varphi_{R+1}(\mathbf{x_0^-})\|^{-2}\left\langle dU_M(\mathbf{x_0^-}), d\varphi_{R+1}(\mathbf{x_0^-})\right\rangle d\varphi_{R+1}(\mathbf{x_0^-}). \end{aligned} \tag{4.51}$$

Using (4.49), we form a Lagrange function

$$\Phi(\mathbf{x}, \mathbf{a}, \mu) = U_M(\mathbf{x}) + \frac{a}{2}\left(\|\mathbf{x}\|^2 - \mathbf{1}\right) - \mu\varphi_{R+1}(\mathbf{x}). \tag{4.52}$$

Since at a conditional extremum point $d\Phi(\mathbf{x}, \mathbf{a}, \boldsymbol{\mu}) = \mathbf{0}$, we have the equality

$$dU_M(\mathbf{e}) = -a\mathbf{e} + \mu d\varphi_{R+1}(\mathbf{e}). \tag{4.53}$$

Taking the scalar product of (4.53) with $\mathbf{e}$ and again applying Euler's theorem, we obtain

$$a = -M \min_{\mathbf{p} \in S^{n-1} \cap \pi} U_M(\mathbf{p}) > 0.$$

On the other hand, taking the scalar product of (4.53) with $d\varphi_{R+1}(\mathbf{e})$, we can find an expression for μ:

$$\mu = \|d\varphi_{R+1}(\mathbf{e})\|^{-2} \langle dU_M(\mathbf{e}), d\varphi_{R+1}(\mathbf{e})\rangle .$$

Therefore, in order to satisfy Eq. (4.51), it is sufficient to set

$$\|\mathbf{x}_0^-\| = \left(\frac{2M\alpha^2}{a}\right)^{\alpha}.$$

Consequently, all the hypothesis of Theorem 1.1.2 is fulfilled and therefore the Eq. (4.48) have a particular solution with the asymptotic expansions

$$\begin{aligned} \mathbf{x}(t) &= \sum\nolimits_{k=0}^{\infty} \mathbf{x}_k \left(\ln(-t)\right) (-t)^{-\alpha(k+2)}, \quad \mathbf{x}_0 = \mathbf{x}_0^-, \\ \mathbf{y}(t) &= \sum\nolimits_{k=0}^{\infty} \mathbf{y}_k \left(\ln(-t)\right) (-t)^{-\alpha(k+M)}, \quad \mathbf{y}_0 = \mathbf{y}_0^-, \\ \sigma(t) &= \sum\nolimits_{k=0}^{\infty} \sigma_k \left(\ln(-t)\right) (-t)^{-\alpha(k-4R)}, \quad \sigma_0 = \sigma_0^-. \end{aligned} \tag{4.54}$$

Now, in order to complete the proof the theorem we need to require that the following equations be satisfied identically:

$$\begin{gathered} g(t) = \left\langle \mathbf{w}\left(\mathbf{x}(t)\right), d_y H\left(\mathbf{x}(t), \mathbf{y}(t)\right)\right\rangle \equiv 0, \\ f(t) = \sigma(t) \left\langle d^2_{yy} H\left(\mathbf{x}(t), \mathbf{y}(t)\right) \mathbf{w}\left(\mathbf{x}(t)\right), \mathbf{w}\left(\mathbf{x}(t)\right)\right\rangle \equiv 1. \end{gathered}$$

The proof of this fact is analogous to arguments used in the proof of Theorem 4.2.4. We first note that as $t \to -\infty$ the scalar functions $g(t)$ and $f(t)$ tend respectively to 0 and 1. We then compute the derivatives of these functions and verify, that they are identically equal to zero.

The theorem is proved.

The theorem as formulated is true also in the case $M = 2$. The proof is given in [63]. The situation in which the matrix $\mathbf{W}(\mathbf{0})$ has a rank that is less than the number of constraints is encountered in investigations of holonomic systems for which the configured manifold has a singularity (e.g. a sharp cone). This kind of example is analyzed in [63].

Based on an analysis of the second variation of the Lagrange functionals (4.45) and (4.52), it is possible to draw conclusions about the properties of *a portion* of the eigenvalues of the Kovalevsky matrix. Unfortunately, this information is insufficient for proving the convergence of the series (4.41) (in the case of Theorem 4.2.4) or (4.54) (in the case of Theorem 4.2.5). Here it must be recalled that the *actual* motions for the systems considered lie on certain invariant manifolds, and thus it is completely possible that, for the determination of the complex structure of particular solutions that have been found, it would be fully adequate to know the properties of just *some* of the eigenvalues of the Kovalevsky matrix. However, in this situation Theorem 1.2.1 is no longer applicable.

In conclusion we note that in the papers [54, 63] a different approach was used for proving the existence of asymptotic trajectories of systems that are restricted by constraints: along with asymptotic expansions of generalized coordinates and momenta, there are constructions of asymptotic expansions of Lagrange multipliers. In particular, it is shown in the paper [63] that if a constraint degenerates at an equilibrium position (here the Lagrange multipliers can be arbitrary) then, on an asymptotic trajectory as $t \to -\infty$, this multiplier will have an unremovable singularity.

4.3 Singular Problems

In this section we will consider a class of problems that were characterized as singular in the preceding chapter, i.e. those in which there is a loss in differentiability due to truncation.

As in Sect. 4.2, we begin with the study of gradient systems (4.33) and introduce yet another assertion—formulated in [106]—concerning a conjecture of R. Thom [67]. In the preceding section we proved a conjecture about asymptotic (as $t \to -\infty$) solutions for system (4.33) with characteristic direction (Theorem 4.2.1) in the totally "degenerate situation" where the expansion of the potential energy in a series in the neighborhood of a critical point begins with terms of at least third order. This theorem is, of course, also true for the case where the absence of a minimum potential can already be discerned in the quadratic terms. This assertion follows from the fact that, in the case considered, the characteristic equation of the first approximation system has *real* positive roots. We now consider a more subtle situation. Assume that the second variation of potential energy, computed at the critical point $\mathbf{x} = \mathbf{0}$, is a positive (but not necessarily positive definite) quadratic form, i.e.

$$U(\mathbf{x}) = \sum_{m=2} U_m(\mathbf{x}), \quad U_2(\mathbf{x}) = \frac{1}{2}\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle \geq \mathbf{0}, \quad \det \mathbf{A} = 0.$$

We consider more closely the operator with symmetric matrix $\mathbf{A}$. Let π be the d–dimensional subspace ($d < n$)

$$\pi = \operatorname{Ker} \mathbf{A} = \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} \colon \mathbf{A}\mathbf{x} = \mathbf{0}\}. \tag{4.55}$$

Theorem 4.3.1 ([106]). *Suppose that the expansion of $U(x)$ has the form*

$$U(\mathbf{x}) = U_2(\mathbf{x}) + \sum_{m=M}^{\infty} U_m(\mathbf{x}), \quad M \geq 3. \tag{4.56}$$

Consider the homogeneous form $V_M(\mathbf{x})$ on the subspace π which is the restriction of $U_M(\mathbf{x})$ to the subspace π ($V_M = U_M|_{\mathbf{x}\in\boldsymbol{\pi}}$). If the form $V_M(\mathbf{x})$ doesn't have a minimum, then the critical point $\mathbf{x} = \mathbf{0}$ of system (4.33) is unstable, so there exists an asymptotic solution $\mathbf{x}(t) \to 0$ as $t \to -\infty$ with characteristic direction (4.34).

Proof. We write (4.33) in the form of a system that is implicit in the derivative:

$$\mathbf{D}(\mathbf{x})\dot{\mathbf{x}} + dU(\mathbf{x}) = \mathbf{0}. \tag{4.57}$$

We represent the vector $\mathbf{x}$ as the sum of two orthogonal components $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, $\mathbf{x}' \in \operatorname{Ker} \mathbf{A} = \boldsymbol{\pi}$, $\mathbf{x}'' \in \operatorname{Im} \mathbf{A} = \boldsymbol{\pi}^{\perp}$, $\mathbb{R}^n = \operatorname{Ker} \oplus \Im \mathbf{A}$. We further let $\mathbf{A}$ denote also the restriction of the operator $\mathbf{A}$ to $\pi^{\perp}$. We can choose a coordinate system for $\mathbb{R}^n$ so that $\mathbf{D}(\mathbf{0}) = \mathbf{E}$. The resulting system, written in the form (4.57), is positive semi-quasihomogeneous in the sense of Definition 3.3.2 with respect to the structure given by the diagonal matrix $\mathbf{G} = \alpha\mathbf{E}$, $\alpha = \frac{1}{M-2}$. Here the parameter β equals α, and the matrix $\mathbf{Q}$ has the form

$$\mathbf{Q} = \operatorname{diag}\big(\underbrace{(M-1)\alpha, \ldots, (M-1)\alpha'}_{x'}, \underbrace{\alpha, \ldots, \alpha}_{x''}\big).$$

We easily see that the truncation of the system is written in terms of two subsystems

$$\dot{\mathbf{x}}' + d_{\mathbf{x}'} U_M(\mathbf{x}', \mathbf{x}'') = \mathbf{0}, \quad \mathbf{A}\mathbf{x}'' = \mathbf{0}. \tag{4.58}$$

To begin, we prove that the system of differential–algebraic equations (4.58) has a particular solution in the form of a ray:

$$\mathbf{x}^-(t) = (-t)^{-\alpha} \mathbf{x}_0^-.$$

Since the form $V_M(\mathbf{x})$, $\mathbf{x} \in \boldsymbol{\pi}$ doesn't have a minimum, we can take $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^{\mathbf{n}}$, $\|\mathbf{e}\| = \mathbf{1}$, is a vector for which the function $U_M(\mathbf{x})$ takes on a minimum value on the sphere $S^{d-1} = \pi \cap S^{n-1} = \{\mathbf{p} \in \mathbb{R}^{\mathbf{n}} \colon \|\mathbf{p}\| = \mathbf{1}, \mathbf{A}\mathbf{p} = \mathbf{0}\}$.

In accordance with (4.55) and taking into account that all eigenvalues of the matrix $\mathbf{A}$ are nonnegative, we consider the Lagrange function

$$\Phi(\mathbf{x}, \mathbf{a}, \mu) = U_M(\mathbf{x}) + \frac{a}{2}\left(\|\mathbf{x}\|^{\mathbf{2}} - \mathbf{1}\right) - \mu U_2(\mathbf{x}). \tag{4.59}$$

We write the condition of the vanishing of the gradient (4.59) for its projections on the subspaces Ker $\mathbf{A}$ and Im$\mathbf{A}$ respectively:

$$\begin{aligned} d_{\mathbf{x}'} U_M(\mathbf{e}, \mathbf{0}) + a\mathbf{e} &= \mathbf{0}, \\ d_{\mathbf{x}''} U_M(\mathbf{e}, \mathbf{0}) - \mu \mathbf{A}\mathbf{e} &= \mathbf{0}. \end{aligned} \tag{4.60}$$

Since $\mathbf{e} \in \boldsymbol{\pi}$, from the second equation of (4.60) it follows that

$$d_{\mathbf{x}''} U_M(\mathbf{x_0^-}) = \|\mathbf{x}_0^-\|^{M-1} d_{\mathbf{x}''} U_M(\mathbf{e}, \mathbf{0}) = \mathbf{0}.$$

On the other hand, since $U_M(\mathbf{x}', \mathbf{0}) = \mathbf{V_M}(\mathbf{x})$, after multiplying the first equation of (4.60) by $\mathbf{e}$ and applying Euler's homogeneous function theorem, we obtain

$$a = -M \min_{\mathbf{p} \in \mathbf{S^{d-1}}} V_M(\mathbf{p}) = -M \min_{\mathbf{p} \in \mathbf{S^{n-1}} \cap \boldsymbol{\pi}} U_M(\mathbf{p}) > 0.$$

Then, using the first equation of (4.59), we find that

$$\|\mathbf{x_0^-}\| = \left(\frac{\alpha}{a}\right)^{\alpha}.$$

The next step in the proof of Theorem 4.3.1 must consist of calculating the matrices $\mathbf{B_k^{(4)}}$ that figure in the hypothesis of Theorem 3.3.1 and the proof of their nonsingularity. It is not difficult to observe that, in the case considered, the matrices

$$\mathbf{B_k^{(4)}} = \mathbf{A}|_{\pi^\perp}, \quad \mathbf{k} = \mathbf{1}, \mathbf{2} \ldots,$$

are indeed nonsingular.

Thus all the hypothesis of Theorem 3.3.1 is satisfied and the system of equations (4.57) (and consequently also those of (4.33)) have a smooth particular solution with asymptotic expansion (4.38).

The theorem is proved.

We remark that the proof given for Theorem 4.3.1 differs slightly from the one presented in [106], since it doesn't depend on the *decomposition lemma* [67].

The next problem we address is related to the converse of the Lagrange theorem on stable equilibrium. The result introduced below was formulated and proved in the article [106]. We revise the proof from [106], which likewise depends on the decomposition lemma.

Theorem 4.3.2. *Let the potential energy of a reversible conservative mechanical system—whose motion is described by the Hamiltonian system of equations (4.3)—have the form (4.56) and suppose that the second variation* $U_2(\mathbf{x}) = \frac{1}{2}\langle A\mathbf{x}, \mathbf{x}\rangle$ *is a positive semi-quasihomogeneous form. Let the form* $V_M(\mathbf{x})$*—where* $V_M(\mathbf{x})$ *is the restriction of the form* $U_M(\mathbf{x})$ *to the subspace* π *defined by the system of equations (4.55)* ($V_M = U_M|_{\mathbf{x}\in\boldsymbol{\pi}}$)*—not have a minimum. Then the equilibrium solution* $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ *of system (4.3) is unstable and there exists a particular asymptotic solution of (4.3):* $(\mathbf{x}(t), \mathbf{y}(t)) \to (\mathbf{0}, \mathbf{0})$ *as* $t \to -\infty$.

Proof. We use Theorem 3.3.2. To this end we rewrite system (4.3) in the explicit form:

$$\dot{\mathbf{y}} = -\boldsymbol{\Psi}(\mathbf{x}, \mathbf{y}) - dU(\mathbf{x}), \quad \dot{\mathbf{x}} = \mathbf{K}(\mathbf{x})\mathbf{y},$$

where $\boldsymbol{\Psi}(\mathbf{x}, \cdot)$ is quadratic in the second argument of the mapping whose coefficients are the partial derivatives of the components of the matrix $\mathbf{K}(\mathbf{x})$.

We project this system on the subspaces π and $\pi^{\perp}$, $\pi \oplus \pi^{\perp} = \mathbb{R}^n$ and, denoting the projection of the vector $\mathbf{y}$ onto the subspace π by $\mathbf{y}'$ and the projection of $\mathbf{y}$ onto the subspace $\pi^{\perp}$ by $\mathbf{y}''$, we write the system as follows:

$$\begin{aligned} \dot{\mathbf{y}}' &= -\boldsymbol{\Psi}'(\mathbf{x}, \mathbf{y}) - d_{\mathbf{x}'} U(\mathbf{x}), \\ \mathbf{x}'' &= -\mathbf{A}^{-1} \left(\dot{\mathbf{y}}'' + \boldsymbol{\Psi}''(\mathbf{x}, \mathbf{y}) + d_{\mathbf{x}''} \left(U(\mathbf{x}) - U_2(\mathbf{x}) \right) \right), \\ \dot{\mathbf{x}}' &= \mathbf{K}'(\mathbf{x})\mathbf{y}' + \mathbf{M}'(\mathbf{x})\mathbf{y}'', \quad \dot{\mathbf{x}}'' = \mathbf{K}''(\mathbf{x})\mathbf{y}'' + \mathbf{M}''(\mathbf{x})\mathbf{y}', \end{aligned} \tag{4.61}$$

where $\boldsymbol{\Psi}', \boldsymbol{\Psi}''$ are the projections of the vector $\boldsymbol{\Psi}$ onto the subspaces π and $\pi^{\perp}$ respectively, where the matrices $\mathbf{K}', \mathbf{K}'', \mathbf{M}', \mathbf{M}''$ have respective orders $d \times d$, $(n-d) \times (n-d)$, $d \times (n-d)$, $(n-d) \times d$, where the matrices $\mathbf{K}'(\mathbf{0}), \mathbf{K}''(\mathbf{0})$ are unitary, and where $\mathbf{M}'(\mathbf{0}), \mathbf{M}''(\mathbf{0})$ are zero matrices of the respective orders (the expansions of the components of $\mathbf{M}'(\mathbf{x}), \mathbf{M}''(\mathbf{x})$ into a Maclaurin series in the neighborhood of $\mathbf{x} = \mathbf{0}$ begin with at least linear terms).

Differentiating the second equation of (4.61) with respect to time, we obtain

$$\dot{x}'' = -A^{-1} \left(\ddot{\mathbf{y}}'' + d_x \boldsymbol{\Psi}''(x, y) + d_y \boldsymbol{\Psi}''(x, y)\dot{y} + d^2_{\mathbf{x}''\mathbf{x}} \left(U(x) - U_2(x) \right) \dot{x} \right) \tag{4.62}$$

We rewrite the last equation of (4.61) in the form

$$y'' = \left(K''(x) \right)^{-1} \left(\dot{x}'' - M''(x) y' \right)$$

and differentiate with respect to t, taking into account (4.62).

Combining the equations obtained with (4.62), and likewise with the first and third groups of the equations from (4.61), we obtain the following system of differential equations:

$$\begin{aligned} \dot{y}' &= -\boldsymbol{\Psi}'(\mathbf{x}, \mathbf{y}) - d_{\mathbf{x}'} U(\mathbf{x}), \\ \dot{y}'' &= -\left(\mathbf{K}''(\mathbf{x}) \right)^{-1} \left(\mathbf{A}^{-1} \dddot{\mathbf{y}}'' + d_{\mathbf{x}} \boldsymbol{\Psi}''(\mathbf{x}, \mathbf{y})\ddot{\mathbf{x}} + d_{\mathbf{y}} \boldsymbol{\Psi}''(\mathbf{x}, \mathbf{y})\ddot{\mathbf{y}} \right) + \\ &\quad + \boldsymbol{\Phi}_1''(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}) + \boldsymbol{\Phi}_2''(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) + \boldsymbol{\Phi}_3''(\mathbf{x}, \mathbf{y}, \dot{\mathbf{y}}) + \boldsymbol{\Phi}_4''(\mathbf{x}, \dot{\mathbf{x}}) + \\ &\quad + \boldsymbol{\Phi}_5''(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}) + \mathbf{P}''(\mathbf{x})\dot{\mathbf{y}}', \\ \dot{\mathbf{x}}' &= \mathbf{K}'(\mathbf{x})\mathbf{y}' + \mathbf{M}'(\mathbf{x})\mathbf{y}'', \\ \dot{\mathbf{x}}'' &= -\mathbf{A}^{-1} (\ddot{y}'' + d_{\mathbf{x}} \boldsymbol{\Psi}''(\mathbf{x}, \mathbf{y})\dot{\mathbf{x}} + d_{\mathbf{y}} \boldsymbol{\Psi}''(\mathbf{x}, \mathbf{y})\dot{\mathbf{y}} + \\ &\quad + d^2_{\mathbf{x}''\mathbf{x}} \left(U(\mathbf{x}) - U_2(\mathbf{x}) \right) \dot{\mathbf{x}}). \end{aligned} \tag{4.63}$$

Here $\mathbf{\Phi}_1''(\mathbf{x}, \mathbf{y}, \cdot)$, $\mathbf{\Phi}_3''(\mathbf{x}, \mathbf{y}, \cdot)$, $\mathbf{\Phi}_4''(\mathbf{x}, \cdot)$ are mappings quadratic in their last argument, $\mathbf{\Phi}_2''(\mathbf{x}, \mathbf{y}, \cdot, \cdot)$, $\mathbf{\Phi}_5''(\mathbf{x}, \cdot, \cdot)$ are mappings bilinear in their last two arguments, and $\mathbf{P}''(\mathbf{x})$ is a matrix whose components have power series expansions in the neighborhood of $\mathbf{x} = \mathbf{0}$ without free terms.

System (4.63), implicit in the higher derivatives, is positive semi-quasihomogeneous in the sense of Definition 3.3.5 with respect to the structure given by the diagonal matrix

$$\mathbf{G} = \operatorname{diag}(\underbrace{M\alpha, \ldots, M\alpha}_{\mathbf{y}}, \underbrace{2\alpha, \ldots, 2\alpha}_{\mathbf{x}}).$$

Here we can set $\beta = \alpha$. Taking into account that the Jacobian matrix $d_{\mathbf{x}}\mathbf{\Psi}''(\mathbf{x}, \cdot)$ is quadratic and that the matrix $d_{\mathbf{y}}\mathbf{\Psi}''(\mathbf{x}, \cdot)$ is linear in the second argument, we compute the corresponding truncation

$$\dot{\mathbf{y}}' + d_{\mathbf{x}'}U_M(\mathbf{x}', \mathbf{x}'') = \mathbf{0}, \quad \dot{\mathbf{y}}'' = \mathbf{0}, \quad \dot{\mathbf{x}}' = \mathbf{y}', \quad \dot{\mathbf{x}}'' = \mathbf{0}. \tag{4.64}$$

We show that the system of differential equations (4.64) has a particular solution in the form of a ray

$$\mathbf{x}^-(t) = (-t)^{-2\alpha}\mathbf{x}_0^-, \ \mathbf{y}^-(t) = (-t)^{-M\alpha}\mathbf{y}_0^-, \ \mathbf{y}_0^- = 2\alpha\mathbf{x}_0^-, \ \mathbf{x}_0^- \in \mathbb{R}^n.$$

Since the form $V_M(\mathbf{x})$, $\mathbf{x} \in \pi$ doesn't have a minimum—then just as in the proof of the preceding theorem—we can set $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^n$, $\|\mathbf{e}\| = 1$ is a vector for which the function $U_M(\mathbf{x})$ attains a minimum value on the set $S^{d-1} = \pi \cap S^{n-1} = \{\mathbf{p} \in \mathbb{R}^n \colon \|\mathbf{p}\| = 1, \mathbf{A}\mathbf{p} = \mathbf{0}\}$.

We consider once more the Lagrange function (4.59) and use the equalities (4.60). From these equalities it follows that

$$\|\mathbf{x}_0^-\| = \left(\frac{2M\alpha^2}{a}\right)^{\alpha}.$$

Thus the truncated system (4.64) possesses the necessary solution. Consequently, in agreement with Theorem 4.3.2, system (4.63) has a particular solution with asymptotic expansion (4.41). System (4.63) is obtained from (4.3) using the differentiation of separate groupings of equations. Integrating these equations from $-\infty$ to t, we obtain that the solutions obtained likewise satisfy (4.3).

The theorem is proved.

We pass to the study of sufficient conditions for the existence of asymptotic solutions to the equations of motion of a mechanical system upon which a nonpotential force is acting. We recall that the motion of such systems, in the most general form, is described by Eq. (4.1). We formulate two variants of a partial converse for Theorems 4.1.2 and 4.1.3 of the first section.

Theorem 4.3.3. *Let* $\det(\mathbf{B}(\mathbf{0})) \neq 0$ *and let* $\mathbf{x} = \mathbf{0}$ *be the only critical point of the homogeneous form* $U_M(\mathbf{x})$*—the first nontrivial form in the expansion of the potential energy* $U(\mathbf{x})$ *in a Maclaurin series. If* M *is odd, then the equilibrium solution* $\mathbf{x} = \mathbf{0}$*,* $\mathbf{y} = \mathbf{0}$ *of the system of equations (4.1) is unstable, and there exists a particular solution of (4.1)* $\mathbf{x}(t) \to \mathbf{0}$*,* $\mathbf{y}(t) \to \mathbf{0}$ *as* $t \to -\infty$*.*

Proof. We rewrite (4.1) in the form

$$\dot{\mathbf{y}} + \boldsymbol{\Psi}(\mathbf{x}, \mathbf{y}) + dU(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{y}, \quad \dot{\mathbf{x}} - \mathbf{K}(\mathbf{x})\mathbf{y} = \mathbf{0}. \tag{4.65}$$

Since $\mathbf{R}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x})$ and $\mathbf{K}(\mathbf{0}) = \mathbf{E}$, we have $\mathbf{R}(\mathbf{0}) = \mathbf{B}(\mathbf{0})$. System (4.65), implicit in the derivatives, is positive semi-quasihomogeneous in the sense of Definition 3.3.2 relative to the structure given by the diagonal matrix

$$\mathbf{G} = \operatorname{diag}\,(\underbrace{(M-1)\alpha, \ldots, (M-1)\alpha}_{\mathbf{y}}, \underbrace{\alpha, \ldots, \alpha}_{\mathbf{x}}).$$

The parameter β should be set equal to α and the matrix $\mathbf{Q}$ should be taken in the form $\mathbf{Q} = (M-1)\alpha\mathbf{E}$. Then the truncation obtained for the system assumes the form

$$dU_M(\mathbf{x}) - \mathbf{B}\mathbf{y} = \mathbf{0}, \quad \dot{\mathbf{x}} - \mathbf{y} = \mathbf{0}, \tag{4.66}$$

with the notation $\mathbf{B} = \mathbf{B}(\mathbf{0})$.

System (4.66) has a particular solution in the form of a quasihomogeneous ray

$$\mathbf{x}^-(t) = (-t)^{-\alpha}\mathbf{x}_0^-, \ \mathbf{y}^-(t) = (-t)^{-(M-1)\alpha}\mathbf{y}_0^-, \ \mathbf{y}_0^- = \alpha\mathbf{x}_0^-, \ \mathbf{x}_0^- \in \mathbb{R}^n.$$

Also, we can set $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^n$, $\|\mathbf{e}\| = 1$ is the unique vector satisfying the equation

$$\mathbf{B}^{-1}dU_M(\mathbf{e}) = a\mathbf{e}$$

for some positive a.

The existence of such a vector follows from Lemma 1.1.1 since, in view of the oddness of M, the index of the homogeneous generalized–gradient vector field $\mathbf{B}^{-1}dU_M(\mathbf{x})$ is even. Consequently, the norm $\|\mathbf{x}_0^-\|$ can be found from the formula

$$\|x_0^-\| = \left(\frac{\alpha}{a}\right)^{\alpha}.$$

So as to be able to use Theorem 3.3.1, it is necessary to compute the matrices $\mathbf{B}_k^{(4)}$, $k = 1, 2, \ldots,$ and to prove their nonsingularity. In the case considered, $\mathbf{B}_k^{(4)} = -\mathbf{B}$, $k = 1, 2, \ldots$, so that the hypothesis of Theorem 3.3.1 is satisfied. This implies that the system of equations (4.65)—and consequently also (4.1)—have a particular solution with asymptotic expansion

$$\mathbf{x}(t) = \sum_{k=0}^{\infty} \mathbf{x}_k\,(\ln(-t))\,(-t)^{-\alpha(k+1)}, \quad \mathbf{x}_0 = \mathbf{x}_0^-,$$
$$\mathbf{y}(t) = \sum_{k=0}^{\infty} \mathbf{y}_k\,(\ln(-t))\,(-t)^{-\alpha(k+M+1)}, \quad \mathbf{y}_0 = \mathbf{y}_0^-, \tag{4.67}$$

and the theorem is proved.

Since, by the oddness of M, the potential energy $U(\mathbf{x})$ of the system necessarily will not have a minimum at $\mathbf{x} = \mathbf{0}$, the given theorem can be regarded as a partial converse of Theorem 4.1.3.

For the case of a gyroscopic force ($\mathbf{B}(\mathbf{x}) \equiv \boldsymbol{\omega}(\mathbf{x})$), Theorem 4.3.3 was first formulated and proved in the scarcely accessible article [57]. Then the theorem was reproved in [25] by constructing the formal invariant manifold. Since in the situation considered there are no dissipative forces, the authors were able to show that this manifold will be symplectic and that the system reduced on it has Hamiltonian form. Still earlier, in [55], the theorem was proved for a system with two degrees of freedom under the condition of constancy for the components of the kinetic energy and gyroscopic force matrices. In the proof, use was made of A.G. Sokol'skiy's theorem on the instability of equilibria of Hamiltonian systems with two degrees of freedom and one null frequency [176]. In its full generality Theorem 4.3.3 was established in [111].

If, conversely, we only deal with dissipative forces, then we can weaken the requirements on the first form in the expansion of potential energy [111].

Theorem 4.3.4. *Let* $\mathbf{B}(\mathbf{x}) = -\mathbf{D}(\mathbf{x})$ *and let* $\mathbf{D}(\mathbf{0})$ *be a positive definite symmetric matrix, so that we have the bound*

$$\langle \mathbf{D}(\mathbf{0})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \geq C\,\|\boldsymbol{\xi}\|^2, \quad \boldsymbol{\xi} \in \mathbb{R}^n, \quad C > 0.$$

If the first nontrivial form $U_M(\mathbf{x})$*,* $M \geq 3$ *in the expansion of the energy of the system in the neighborhood of* $\mathbf{x} = \mathbf{0}$ *does not have a minimum, then the equilibrium position* $\mathbf{x} = \mathbf{0}$*,* $\mathbf{y} = \mathbf{0}$ *of system (4.1) is unstable and there exists a particular asymptotic solution* $\mathbf{x}(t) \to \mathbf{0}$*,* $\mathbf{y}(t) \to \mathbf{0}$ *of system (4.1) as* $t \to -\infty$*.*

Proof. Only in a few details is the proof different from that of the preceding theorem. A solution of the truncated system (4.66) in the form of a ray

$$\mathbf{x}^-(t) = (-t)^{-\alpha}\mathbf{x}_0^-,\ \mathbf{y}^-(t) = (-t)^{-(M-1)\alpha}\mathbf{y}_0^-,\ \mathbf{y}_0^- = \alpha\mathbf{x}_0^-,\ \mathbf{x}_0^- \in \mathbb{R}^n$$

is constructed in the following way: we set $\mathbf{x}_0^- = \nu\mathbf{p}$, where $\mathbf{p} \in \mathbb{R}^n$, $\mathbf{p} \neq 0$, is a vector for which $U_M(x)$ assumes its minimum value on the ellipsoid

$$S^{n-1} = \{p \in \mathbb{R}^n\colon \langle \mathbf{D}\mathbf{p}, \mathbf{p}\rangle = 1\}.$$

Here we have retained the notation S^{n-1}, since an ellipsoid is topologically equivalent to a sphere. Here likewise $\mathbf{D}(\mathbf{0}) = \mathbf{B}(\mathbf{0}) = \mathbf{B}$. We consider the Lagrange function

$$\boldsymbol{\Phi}(\mathbf{x}, a) = U_M(\mathbf{x}) + \frac{a}{2}\,(\langle \mathbf{D}\mathbf{x}, \mathbf{x}\rangle - 1). \tag{4.68}$$

The conditions for an extremum of the function (4.68) lead to the equation

$$dU_M(\mathbf{p}) = -a\mathbf{D}\mathbf{p}. \tag{4.69}$$

Taking the scalar product of (4.69) with p, we obtain the inequality

$$a = -M \min_{p \in S^{n-1}} U_M(\mathbf{p}) > 0.$$

It is likewise easy to see that

$$\nu = \left(\frac{\alpha}{a}\right)^{\alpha}.$$

Then, applying Theorem 3.3.1, we obtain that the Eq. (4.1) have a particular solution with asymptotic (4.67).

The theorem is proved.

In a yet earlier paper [58], a problem was considered that is in a certain sense "opposite" to the one we have considered: sufficient conditions were studied for the existence of trajectories of the system that are placed under the influence of dissipative forces and that correspond to solutions of equations of motion that are asymptotic to equilibria as $t \to +\infty$. It is interesting that for the existence of such solutions it turned out to be sufficient that $U_M(x)$ not have a maximum.

We also note that Theorem 4.3.4 also remains true in the case $M = 2$. For the proof it suffices to consider the system of first approximation.

In conclusion we consider, for a system of equations of type (4.7), the problem of finding sufficient conditions for the instability of the trivial equilibrium solution and the existence of asymptotic solutions as $t \to -\infty$. In Sect. 4.1 we proved a theorem on the asymptotic stability of an equilibrium solution of system (4.7). Although the conditions for asymptotic stability seem to be almost unimprovable, the conditions for instability can be considerably weakened. In particular, below we will no longer demand that all the eigenvalues of the matrix $\mathbf{\Lambda} = d_c f(\mathbf{0}, \mathbf{0}, \mathbf{0})$ lie in the left half-plane. We are limited only to requiring the nonsingularity of this matrix.

Theorem 4.3.5. *Let* $\det \mathbf{B} \neq 0$, $\det \mathbf{\Lambda} \neq 0$, *where* $\mathbf{B} = \mathbf{B}(\mathbf{0}, \mathbf{0})$, *and let* $\mathbf{x} = \mathbf{0}$ *be the only critical point of the first nontrivial form* $U_M(\mathbf{x}, \mathbf{0})$ *in the expansion of potential energy with odd* M. *Then the equilibrium solution* $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$ *of the system of equations (4.7) is unstable, and there exists an asymptotic solution of system (4.7)* $(\mathbf{x}(t), \mathbf{y}(t), \mathbf{c}(t)) \to (\mathbf{0}, \mathbf{0}, \mathbf{0})$ *as* $t \to -\infty$.

Proof. We rewrite (4.7) in the form of a system of equations that is implicit in the derivatives:

$$\begin{gathered}\dot{\mathbf{y}} + \mathbf{\Psi}(\mathbf{x}, \mathbf{y}, \mathbf{c}) + dU(\mathbf{x}, \mathbf{c}) - \mathbf{R}(\mathbf{x}, \mathbf{c})\mathbf{y} = \mathbf{0}, \\ \dot{\mathbf{x}} - \mathbf{K}(\mathbf{x}, \mathbf{c})\mathbf{y} = \mathbf{0}, \quad \dot{\mathbf{c}} - \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{c}) = \mathbf{0},\end{gathered} \tag{4.70}$$

where $\mathbf{\Psi}(\mathbf{x}, \cdot, \mathbf{c})$ is a mapping that is quadratic in its second argument.

As was stipulated earlier, $\mathbf{K}(\mathbf{0},\mathbf{0}) = \mathbf{E}$, so that

$$\mathbf{R}(\mathbf{0},\mathbf{0}) = \mathbf{B}(\mathbf{0},\mathbf{0}) = \mathbf{B}.$$

System (4.70) is positive semi-quasihomogeneous in the sense of Definition 3.3.2 with respect to the structure given by the diagonal matrix

$$G = \operatorname{diag}\,(\underbrace{(M-1)\alpha,\ldots,(M-1)\alpha}_{y},\underbrace{\alpha,\ldots,\alpha}_{x},\underbrace{\alpha,\ldots,\alpha}_{c}).$$

Here we can set $\beta = \alpha$, so that the matrix $\mathbf{Q}$ equals the diagonal matrix $\mathbf{G}$, whose diagonal entries are then

$$\underbrace{(M-1)\alpha,\ldots,(M-1)\alpha}_{y},\underbrace{(M-1)\alpha,\ldots,(M-1)\alpha}_{x},\underbrace{\alpha,\ldots,\alpha}_{c}).$$

We write the truncated system:

$$dU_M(\mathbf{x},\mathbf{0}) - \mathbf{B}\mathbf{y} = \mathbf{0}, \quad \dot{\mathbf{x}} - \mathbf{y} = \mathbf{0}, \quad \mathbf{\Lambda}\mathbf{c} + \mathbf{\Lambda}_{\mathbf{x}}\mathbf{x} = \mathbf{0}, \tag{4.71}$$

where $\mathbf{\Lambda}_{\mathbf{x}} = d_x f(\mathbf{0},\mathbf{0},\mathbf{0})$. In addition to fulfilling the hypothesis of the theorem, this system has a particular solution in the form of a quasihomogeneous ray:

$$\mathbf{x}^-(t) = (-t)^{-\alpha}\mathbf{x}_0^-, \quad \mathbf{y}^-(t) = (-t)^{-(M-1)\alpha}\mathbf{y}_0^-, \quad \mathbf{c}^-(t) = (-t)^{-\alpha}\mathbf{c}_0^-,$$

$$\mathbf{y}_0^- = \alpha\mathbf{x}_0^-, \quad \mathbf{x}_0^- \in \mathbb{R}^n, \quad \mathbf{c}_0^- \in \mathbb{R}^d.$$

From the third group of equations in (4.71), it follows that

$$\mathbf{c}_0^- = -\mathbf{\Lambda}^{-1}\mathbf{\Lambda}_{\mathbf{x}}\mathbf{x}_0^-.$$

As in the proof of Theorem 4.3.3 we can set $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$, where $\mathbf{e}$ is a unit vector—whose existence follows from Lemma 1.1.1—satisfying the equality

$$\mathbf{B}^{-1}\,dU_M(\mathbf{e},\mathbf{0}) = a\mathbf{e}$$

for some positive a, and thus

$$\|\mathbf{x}_0^-\| = \left(\frac{\alpha}{a}\right)^{\alpha}.$$

The matrices $\mathbf{B}_k^{(4)}$, $k = 1, 2, \ldots,$ figuring in the formulation of Theorem 3.3.1 have the block form

$$\begin{pmatrix} -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}$$

and, by the hypothesis of the theorem, their determinants are nonzero.

Thus all conditions of Theorem 3.3.1 are fulfilled, whereby system (4.70)—and consequently also (4.7)—has a particular solution with asymptotic expansion

$$
\begin{aligned}
\mathbf{x}(t) &= \sum_{k=0}^{\infty} \mathbf{x}_k \, (\ln(-t)) \, (-t)^{-\alpha(k+1)}, & \mathbf{x}_0 &= \mathbf{x}_0^-, \\
\mathbf{y}(t) &= \sum_{k=0}^{\infty} \mathbf{y}_k \, (\ln(-t)) \, (-t)^{-\alpha(k+M-1)}, & \mathbf{y}_0 &= \mathbf{y}_0^-, \\
\mathbf{c}(t) &= \sum_{k=0}^{\infty} \mathbf{c}_k \, (\ln(-t)) \, (-t)^{-\alpha(k+1)}, & \mathbf{c}_0 &= \mathbf{c}_0^-,
\end{aligned}
\tag{4.72}
$$

and the theorem is proved.

If no gyroscopic forces act on the system, then the requirements placed on the forms $U_M(\mathbf{x}, \mathbf{0})$ can be weakened, as was the case also with Theorems 4.3.3 and 4.3.4.

Theorem 4.3.6. *Let* $\mathbf{B}(\mathbf{x}, \mathbf{c}) \equiv -\mathbf{D}(\mathbf{x}, \mathbf{c})$*, and let* $\mathbf{D}(\mathbf{0}, \mathbf{0})$ *be a positive definite symmetric matrix, i.e. we have the bound*

$$\langle \mathbf{D}(\mathbf{0}, \mathbf{0})\boldsymbol{\xi}, \boldsymbol{\xi} \rangle \geq C \|\boldsymbol{\xi}\|^2, \quad \boldsymbol{\xi} \in \mathbb{R}^n, C > 0.$$

If $\det \boldsymbol{\Lambda} \neq 0$*, and for* $\mathbf{c} = \mathbf{0}$*, the first nontrivial form* $U_M(\mathbf{x}, 0)$*,* $M \geq 3$*, of the expansion of the potential energy in a Maclaurin series in the neighborhood of* $\mathbf{x} = \mathbf{0}$ *doesn't have a minimum, then the equilibrium solution* $\mathbf{x} = \mathbf{0}$*,* $\mathbf{y} = \mathbf{0}$*,* $\mathbf{c} = \mathbf{0}$ *of Eq. (4.7) is unstable, and there exists an asymptotic solution of system (4.7) as* $t \to -\infty$*:* $(\mathbf{x}(t), \mathbf{y}(t), \mathbf{c}(t)) \to (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

The proof of this theorem combines elements from the proofs of Theorems 4.3.4 and 4.3.5. To construct a solution of system (4.71) in the form of a ray, we take $\mathbf{x}_0^- = \nu \mathbf{p}$, where $\mathbf{p} \in \mathbb{R}^n$, $\mathbf{p} \neq \mathbf{0}$, is the vector for which $U_M(\mathbf{x}, \mathbf{0})$ takes on its minimum value on the ellipsoid

$$S^{n-1} = \{\mathbf{p} \in \mathbb{R}^n : \langle \mathbf{D}\mathbf{p}, \mathbf{p} \rangle = 1\},$$

where $\mathbf{D} = \mathbf{D}(\mathbf{0}, \mathbf{0})$ and

$$\nu = \left(\frac{\alpha}{a}\right)^{-\alpha}.$$

The remainder of the argument is a repetition of the proof of Theorem 4.3.5. The expansions of asymptotic solutions into series again have the form (4.72).

The theorem is proved.

It turns out that the instability of the equilibrium solution for (4.7) under the absence of a minimum of the form $U_M(x, 0)$ is preserved when only potential forces act on the system.

Theorem 4.3.7. *Suppose that* $\mathbf{B}(\mathbf{x}, \mathbf{c}) \equiv \mathbf{0}$*,* $\det \boldsymbol{\Lambda} \neq 0$ *and that the first nontrivial form* $U_M(\mathbf{x}, \mathbf{0})$*,* $M \geq 3$ *in the expansion of potential energy with* $\mathbf{c} = \mathbf{0}$ *into a Maclaurin series in the neighborhood of* $\mathbf{x} = \mathbf{0}$ *doesn't have a minimum. Then the solution* $\mathbf{x} = \mathbf{0}$*,* $\mathbf{y} = \mathbf{0}$*,* $\mathbf{c} = \mathbf{0}$ *of system (4.7) is unstable and there exists an* *asymptotic solution* $(\mathbf{x}(t), \mathbf{y}(t), \mathbf{c}(t))$ *of system (4.7) which approaches* $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ *as* $t \to -\infty$.

Proof. Although superficially Theorem 4.3.7 appears to be only a special case of Theorem 4.3.6, there is a fundamental difference between these two assertions: the absence of nonpotential forces leads to a different quasihomogeneous scale and, consequently, to a different expansion of asymptotic solutions.

Again we write system (4.7) as a system implicit in the derivatives:

$$\begin{gathered}\dot{\mathbf{y}} + \boldsymbol{\Psi}(\mathbf{x}, \mathbf{y}, \mathbf{c}) + dU(\mathbf{x}, \mathbf{c}) = \mathbf{0}, \\ \dot{\mathbf{x}} - \mathbf{K}(\mathbf{x}, \mathbf{c})\mathbf{y} = \mathbf{0}, \;\; \dot{\mathbf{c}} - \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{c}) = \mathbf{0}.\end{gathered} \tag{4.73}$$

System (4.73) is positive semi-quasihomogeneous in the sense of Definition 3.3.2 with respect to the structure given by the diagonal matrix

$$\mathbf{G} = \operatorname{diag}(\underbrace{M\alpha, \dots, M\alpha}_{y}, \underbrace{2\alpha, \dots, 2\alpha}_{x}, \underbrace{2\alpha, \dots, 2\alpha}_{c}).$$

Here $\beta = \alpha$ and the matrix $\mathbf{Q}$ has the form

$$\mathbf{Q} = \operatorname{diag}(\underbrace{2(M-1)\alpha, \dots, 2(M-1)\alpha}_{y}, \underbrace{M\alpha, \dots, M\alpha}_{x}, \underbrace{2\alpha, \dots, 2\alpha}_{c}).$$

It is easy to see that in this case the truncated system has the appearance

$$\dot{\mathbf{y}} + dU_M(\mathbf{x}, \mathbf{0}) = \mathbf{0}, \quad \dot{\mathbf{x}} - \mathbf{y} = \mathbf{0}, \quad \boldsymbol{\Lambda} + \boldsymbol{\Lambda}_{\mathbf{x}}\mathbf{x} = \mathbf{0}. \tag{4.74}$$

and that it has a particular solution in the form of a quasihomogeneous ray:

$$\begin{gathered}\mathbf{x}^-(t) = (-t)^{-2\alpha}\mathbf{x}_0^-, \quad \mathbf{y}^-(t) = (-t)^{-M\alpha}\mathbf{y}_0^-, \quad \mathbf{c}^-(t) = (-t)^{-2\alpha}\mathbf{c}_0^-, \\ \mathbf{y}_0^- = 2\alpha\mathbf{x}_0^-, \quad \mathbf{x}_0^- \in \mathbb{R}^n, \quad \mathbf{c}_0^- \in \mathbb{R}^d.\end{gathered}$$

From the third group of equations of (4.74) it follows that

$$c_0^- = -\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{\mathbf{x}}\mathbf{x}_0^-.$$

We set $\mathbf{x}_0^- = \|\mathbf{x}_0^-\|\mathbf{e}$, where $\mathbf{e}$ is the unit vector for which the form $U_M(\mathbf{x}, \mathbf{0})$ assumes its minimum value on the sphere

$$S^{n-1} = \{\mathbf{p} \in \mathbb{R}^n \colon \|\mathbf{p}\| = 1\}.$$

In addition, as before (in Theorems 4.2.3, 4.2.5 and 4.3.2),

$$\|\mathbf{x}_0^-\| = \left(\frac{2M\alpha^2}{a}\right)^{\alpha}.$$

Further, using the algorithm of Theorem 3.3.1, we compute the matrices $\mathbf{B}_k^{(4)}$, $k = 1, 2, \ldots$. In the case considered, these matrices equal $\mathbf{\Lambda}$. Consequently, in view of the nonsingularity of $\mathbf{\Lambda}$, all the hypothesis of Theorem 3.3.1 is satisfied and system (4.73), and consequently also (4.7), has a particular solution whose components have the asymptotic expansions

$$
\begin{aligned}
\mathbf{x}(t) &= \sum\nolimits_{k=0}^{\infty} \mathbf{x}_k\,(\ln(-t))\,(-t)^{-\alpha(k+2)}, && \mathbf{x}_0 = \mathbf{x}_0^-,\\
\mathbf{y}(t) &= \sum\nolimits_{k=0}^{\infty} \mathbf{y}_k\,(\ln(-t))\,(-t)^{-\alpha(k+M)}, && \mathbf{y}_0 = \mathbf{y}_0^-\\
\mathbf{c}(t) &= \sum\nolimits_{k=0}^{\infty} \mathbf{c}_k\,(\ln(-t))\,(-t)^{-\alpha(k+2)}, && \mathbf{c}_0 = \mathbf{c}_0^-.
\end{aligned}
\tag{4.75}
$$

The theorem is proved.

As we see, the series (4.72) and (4.75) differ. This is connected with the fact that system (4.74), in comparison with (4.71), has lost less in way of derivatives and for this reason the properties of the original solutions of system (4.7), in the presence and absence of potential forces, must differ.

To conclude this section we introduce some new results concerning the impossibility of gyroscopic stabilization of singular equilibrium positions for a mechanical system. We first consider the strongly singular case where the Maclaurin series for the potential energy begins with a homogeneous form of degree $m \geq 3$:

$$U = U_m + U_{m+1} + \ldots.$$

We will suppose that $\mathbf{x} = \mathbf{0}$ is the only critical point of the homogeneous polynomial U_m. In particular, the equilibrium point $\mathbf{x} = \mathbf{0}$ will be isolated.

We also suppose that the equilibrium point isn't an extreme point for the form U_m. In this case the cone

$$\{\mathbf{x} \in \mathbb{R}^n : U_m(\mathbf{x}) = 0\} \tag{4.76}$$

doesn't reduce to the single point $\mathbf{x} = \mathbf{0}$. Let $\mathbf{\Lambda}$ be the intersection of the cone (4.76) with the sphere

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1\}.$$

Under the conditions just stated $\mathbf{\Lambda}$ is a regular algebraic manifold of dimension $n-2$. It may consist of several connected components.

We furthermore set $\mathbf{\Gamma} = \mathbf{\Omega}|_{x=0}$, so that $\mathbf{\Gamma}$ is a skew-symmetric $n \times n$ matrix.

Theorem 4.3.8. *Suppose that* $\det \mathbf{\Gamma} \neq 0$ *and that the Euler characteristic of* $\mathbf{\Lambda}$ *is different from zero. Then the equilibrium point* $\mathbf{x} = \mathbf{0}$ *is unstable.*

The manifold $\mathbf{\Lambda}$ is, by the way, invariant under the involution $\mathbf{x} \mapsto -\mathbf{x}$. This means that we can also consider the manifold in the projective space $\mathbb{R}P^{n-1}$, which is obtained from the sphere S^{n-1} by identification of antipodal points. The study of the topology of real projective manifolds is a completely classical problem. If the Euler characteristic $\chi(\mathbf{\Lambda})$ is nonzero, then this condition is satisfied by at least one connected component of $\mathbf{\Lambda}$.

Since the skew-symmetric matrix $\mathbf{\Gamma}$ is nonsingular, n must be even. Therefore $\dim \mathbf{\Lambda}$ is likewise even. Recall that the Euler characteristic of any of any odd-dimensional compact manifold is zero.

We sketch the proof of Theorem 4.3.8, where we represent the equation of motion in a form that is implicit in the higher derivatives:

$$\mathbf{\Gamma}\dot{\mathbf{x}} = -\frac{\partial U_m}{\partial \mathbf{x}} + \mathbf{G}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}).$$

Here $\mathbf{G}$ is a smooth vector function that reduces to zero for $\mathbf{x} = \dot{\mathbf{x}} = \ddot{\mathbf{x}} = \mathbf{0}$. We choose a nonlinear Hamiltonian first order system

$$\mathbf{\Gamma}\dot{\mathbf{x}} = -\frac{\partial U_m}{\partial \mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^n$$

and we will look for its nontrivial power solution

$$\mathbf{x}(t) = \frac{\mathbf{c}}{t^\alpha}, \quad \mathbf{c} \in \mathbb{R}^n \backslash \{\mathbf{0}\}, \quad \alpha = \frac{1}{m-2} > 0. \tag{4.77}$$

The result depends on the presence of solutions of the algebraic system of equations

$$\alpha \mathbf{c} = \mathbf{\Gamma}^{-1} \frac{\partial U_m}{\partial \mathbf{x}}|_{x=c}. \tag{4.78}$$

In the case considered, Lemma 1.1.1 is, unfortunately, inapplicable. The solvability of system (4.78) derives from other topological considerations.

The trajectories of solutions of type (4.77) obviously lie on the cone (4.76).

Let $\mathbf{w}$ be the vector field on Λ^{n-2} that is obtained by projection of vectors $\mathbf{\Gamma}^{-1}U_m^1(\mathbf{x})$, $\mathbf{x} \in \mathbf{\Lambda}^{n-2}$ onto the tangent subspace $T_x\mathbf{\Lambda}$. Since $\chi(\mathbf{\Lambda}) \neq 0$, this field has critical points at which

$$\lambda \mathbf{c} = \mathbf{\Gamma}^{-1} \frac{\partial U_m}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{c}}, \quad \langle \mathbf{c}, \mathbf{c} \rangle = 1,$$

and $\lambda \neq 0$. Multiplying the vector $\mathbf{c}$ by a suitable number, we obtain a nontrivial solution of the algebraic equation (4.78) (possibly with a minus sign on the left side). But this indicates the presence in the truncated system of an “entering” or “exiting” ray. According to Theorem 3.1.1, the full system of equations of motion enjoys the very same property. Since the original system is Hamiltonian, its phase flow preserves standard measure in phase space. If there is an asymptotic solution that “enters” the equilibrium position, then this equilibrium is likewise unstable according to Lemma 1.3.4.

In the general case, following the *decomposition lemma* [67], in the neighborhood of the critical point $\mathbf{x} = \mathbf{0}$ the potential energy reduces to

$$\frac{1}{2}(\pm x_1^2 \pm \ldots \pm x_r^2) + W(x_{r+1}, \ldots, x_n),$$

where W is a smooth function in $n - r$ variables $\mathbf{z} = (x_{r+1}, \ldots, x_n)$, so that its Maclaurin series begins with a form of order $k \geq 3$:

$$W = W_k + W_{k+1} + \ldots.$$

Let $\boldsymbol{\Pi} = \{\mathbf{z}\}$ be the $(n-r)$–dimensional linear space determined by the equations $x_1 = \ldots = x_r = 0$. We will suppose that $\mathbf{z} = \mathbf{0}$ is the only critical point of the homogeneous function $W_k : \boldsymbol{\Pi} \to \mathbb{R}$. In particular, the equilibrium $\mathbf{x} = \mathbf{0}$ of the system considered will be isolated. If the form W_k takes on opposite signs, then

$$\tilde{\boldsymbol{\Lambda}} = \{\mathbf{z} \in \boldsymbol{\Pi} : W_k(\mathbf{z}) = 0, \quad \langle \mathbf{z}, \mathbf{z} \rangle = 1\}$$

is a nonempty regular manifold of dimension $n - r - 2$.

We bound the two–form of gyroscopic forces

$$(\boldsymbol{\Gamma}\mathbf{x}', \mathbf{x}'') \tag{4.79}$$

on the linear subspace $\boldsymbol{\Pi}$. This bound will be a nondegenerate two-form if for any $\mathbf{x}' \in \boldsymbol{\Pi} \setminus \{\mathbf{0}\}$ there is an $\mathbf{x}'' \in \boldsymbol{\Pi}$ such that the value of (4.79) is nonzero. For nondegeneracy it is necessary that $n - r$ be even. The criterion for nondegeneracy is that the skew-symmetric matrix $\tilde{\boldsymbol{\Gamma}}$—the intersection of the last $n - r$ rows and $n - r$ columns of the matrix $\boldsymbol{\Gamma}$—is nonsingular.

Theorem 4.3.9 ([114]). *Suppose that* $\det \tilde{\boldsymbol{\Gamma}} \neq 0$ *and that one of the following holds:*

1. *The index of the gradient vector field* $\mathbf{z} \mapsto W_k'(\mathbf{z})$ *at the point* $\mathbf{z} = \mathbf{0}$ *is different from* $(-1)^{n-r}$,
2. $\chi(\tilde{\boldsymbol{\Lambda}}) \neq 0$.

Then the equilibrium point $\mathbf{x} = \mathbf{0}$ *is unstable.*

This assertion is proved in the same way as Theorem 4.3.8. The equations of motion are written in the following form:

$$\mathbf{y} = \ldots, \quad \tilde{\boldsymbol{\Gamma}}\dot{\mathbf{z}} = -\frac{\partial W_k}{\partial \mathbf{z}} + \ldots,$$

where $\mathbf{y} = (x_1, \ldots, x_r)$ and where the dots in the display represent a series of monomials in the variables $\mathbf{x}, \dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$. Conditions (1) and (2) guarantee that the Hamiltonian system

$$\tilde{\mathbf{\Gamma}}\dot{\mathbf{z}} = -\frac{\partial W_k}{\partial \mathbf{z}} \tag{4.80}$$

has an asymptotic solution of the form (4.80), which then is successively "torn apart" in the formal series in inverse powers of time, which satisfies the "full" system of equation (4.80).

To summarize, we note that equations of motion, having asymptotic solutions of nonexponential order, converging to the equilibrium position, are singular if their linearization is nontrivial, i.e. if the corresponding characteristic equation has nonzero roots. From the mechanical point of view we add to this effect:

(a) The nontriviality of the second variation of the potential energy at the equilibrium position;
(b) The nonsingularity of the matrix determining the nonpotential forces, calculated at the equilibrium position;
(c) Change over time of the system parameters.

In all these cases the asymptotic series, constructed for the corresponding solutions, will most likely diverge.

Appendix A
Nonexponential Asymptotic Solutions of Systems of Functional-Differential Equations

As was mentioned in the introduction, the methods presented for constructing particular solutions are valid for a wider class of objects than systems of ordinary differential equations, among the simplest of which are systems of differential equations with deviating arguments:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{x}(t+t_1), \ldots, \mathbf{x}(t+t_s)\right), \ \mathbf{x} \in \mathbb{R}^n, \ t_1, \ldots, t_s \in \mathbb{R}, \tag{A.1}$$

where $\mathbf{f} = \mathbf{f}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right)$ is a smooth vector function of its arguments, where we will suppose that each of its components can be expanded in a Maclaurin series in the component vectors $\mathbf{x}_{(0)} = \mathbf{x}(t), \ldots, \mathbf{x}_{(s)} = \mathbf{x}(t+t_s)$.

For an acquaintance with the theory of such systems we can recommend the monograph [47]. We suppose that $\mathbf{x}(t) \equiv \mathbf{0}$ is a trivial solution of the system (A.1), i.e. $\mathbf{f}(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}$. Then the question of the stability of such a solution arises entirely naturally. The Lyapunov theory of stability with some (occasionally very substantive) changes translates to systems with deviating arguments. We won't dwell on the details and refer to the monograph [47] just cited. N.N. Krasovskiy proved theorems about stability and instability with respect to the first approximation (see his original paper [123] or see [47]). These theorems were mainly proved by means of the *second* Lyapunov method, which is likewise used for the analysis of critical cases [165–168]. It should be noted right away that the analysis of critical cases for systems of equations with a deviating argument is a highly laborious problem and that stability criteria are practically never expressed in terms of the coefficients of the *original* system. Thus the classical ideas of Lyapunov's first method is rather rarely used in the theory of such systems. We show that these ideas are useful in obtaining necessary conditions for instability precisely in the critical cases. For systems of the type considered, an important theorem on the center manifold was proved by Yu.S. Osipov under additional assumptions [147], which allowed the connection of investigations of the critical cases of stability of systems of equations with deviating argument with the analysis of certain finite-dimensional systems of differential equations. We will consider "supercritical" cases, where all

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the roots of the first approximation system are zero and, consequently, reduction on the finite-dimensional center manifold is impossible.

We equip the space $(\mathbb{R}^n)^{s+1}$ of variables $\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$ with a quasihomogeneous structure. As usual, let $\mathbf{G}$ be some matrix with real elements, whose eigenvalues have strictly *positive* real parts. We represent the group of quasihomogeneous dilations in the following form:

$$\mathbf{x}_{(0)} \mapsto \mu_{(0)}^{\mathbf{G}}\mathbf{x},\ \mathbf{x}_{(1)} \mapsto \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mathbf{x}_{(s)} \mapsto \mu_{(s)}^{\mathbf{G}}\mathbf{x}.$$

Definition A.1. We say that the system of equations (A.1) is *quasihomogeneous* with respect to the quasihomogeneous structure generated by the matrix $\mathbf{G}$, and we denote its right side by $\mathbf{f} = \mathbf{f}_{\mathrm{q}}$ if, for any $\mathbf{x}_{(0)}$, $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$ and any $\mu \in \mathbb{R}^+$, the following equality is satisfied:

$$\mathbf{f}_{\mathrm{q}}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}\right) = \mu^{\mathbf{G}+\mathbf{E}}\mathbf{f}_{\mathrm{q}}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right). \tag{A.2}$$

It should be noted that, although for quasihomogeneous systems of ordinary differential equations the quasihomogeneous structure itself gives rise to a particular solution of ray type, quasihomogeneous systems of equations with deviating arguments generally don't have such solutions.

Definition A.2. We call the system of equations (A.1) *semi-quasihomogeneous* with respect to the structure generated by the matrix $\mathbf{G}$ if its right side can be represented as a formal sum

$$\mathbf{f}_{\mathrm{q}}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right)$$

such that there exists a positive number β such that, for any

$$\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$$

and for any $m = 0, 1, 2, \ldots$, the form $\mathbf{f}_{\mathrm{q}+m}$ satisfies the equality:

$$\mathbf{f}_{\mathrm{q}+m}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}\right) \mu^{\mathbf{G}+m\beta\mathbf{E}}\mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right). \tag{A.3}$$

By virtue of the specific systems of equations of type (A.1), it makes sense only to consider positive semi-quasihomogeneous systems, so that everywhere in the sequel we will assume that $\beta > 0$.

The selection of quasihomogeneous truncations of systems of equations with a deviating argument can be realized with the aid of the already described technique of Newton manifolds, which we will amply employ.

We consider a model system of *ordinary* differential equations:

$$\dot{\mathbf{x}} = \mathbf{g}_{\mathrm{q}}(\mathbf{x}), \tag{A.4}$$

where

$$\mathbf{g}_{\mathrm{q}}(\mathbf{x}) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, \dots, \mathbf{x}).$$

It turns out that, in the "supercritical" case, a deviating argument in fact has no effect on the *instability* of the system. Roughly speaking, if system (A.4) is unstable, so too will system (A.1) be unstable. More precisely, the following assertion holds, which generalizes a theorem in the article [64].

Theorem A.1. *Let the system (A.1) be semi-quasihomogeneous and suppose that all the following conditions hold:*

1. *There exist a vector* $\mathbf{x}_0^\gamma \in \mathbb{R}^n$, $\mathbf{x}_0^\gamma \neq \mathbf{0}$ *and a number* $\gamma = \pm 1$ *such that the following equality holds:*

$$-\gamma \mathbf{G}\mathbf{x}_0^\gamma = \mathbf{g}_{\mathrm{q}}(\mathbf{x}_0^\gamma), \tag{A.5}$$

2. $$\operatorname{sign}\gamma = \operatorname{sign} t_j, \quad j = 1, \dots, s. \tag{A.6}$$

Then system (A.1) has a particular solution $\mathbf{x}(t) \to 0$ *as* $St \to \gamma \times \infty$.

Proof.

First step. Construction of a formal solution.

We will look for the solution in the customary form:

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta}. \tag{A.7}$$

For proving the existence of a formal solution of system (A.1) in the form (A.7), we apply Theorem 3.3.2. Let $t_j \neq 0$ be one of the numbers $t_1, \dots, t_s$. We consider the formal expansion of the vector function $\mathbf{x}_{(j)}(t) = \mathbf{x}(t + t_j)$ in powers of the quantity t_j:

$$\mathbf{x}_{(j)}(t) = \sum_{p=0}^{\infty} \frac{\mathbf{x}^{(p)}(t)}{p!} (t_j)^p.$$

From this expansion it is clear that from the formal point of view the system of equations (A.1) can be rewritten in the form of the following system of ordinary differential equations

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}}\left(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), \dots, \mathbf{x}^{(p)}(t), \dots\right), \tag{A.8}$$

whose right side contains an *unlimited* number of higher order derivatives.

System (A.8), implicit in (and generally unsolvable for) the higher derivatives, is positive semi-quasihomogeneous with respect to the structure given by the matrix $\mathbf{G}$ in the sense of Definition 3.3.5. Its quasihomogeneous truncation clearly coincides with the system of ordinary differential equations (A.4) (see Eqs. (A.2) and (A.3)) which, in view of (A.5), has a solution in the form of a quasihomogeneous ray. Therefore the existence of a formal particular solution of system (A.1) in the form (A.7) automatically follows from Theorem 3.3.2.

It should be noted that we did not use condition (A.6) for constructing a *formal* asymptotic solution for the system considered.

Second step. The proof of existence of an asymptotic solution for system (A.1), for which (A.7) is an asymptotic expansion.

We make a change of dependent and independent variables by the formulas:

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(\xi), \quad \xi = \varepsilon^{-1}(\gamma t)^{-\beta}, \quad 0 < \varepsilon \ll 1.$$

After this, system (A.1) assumes the form:

$$-\gamma\beta\xi\frac{d\mathbf{y}}{d\xi}(\xi) = \gamma\mathbf{G}\mathbf{y}(\xi) + \sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{\mathrm{q}+m}\left(\mathbf{y}_{(0)}(\xi,\varepsilon),\ldots,\mathbf{y}_{(s)}(\xi,\varepsilon)\right), \tag{A.9}$$

where we have introduced the following notation:

$$\mathbf{y}_{(j)}(\xi,\varepsilon) = \left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\mathbf{G}}\mathbf{y}\left(\xi\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right),$$
$$j = 0,\ldots,s, \quad t_0 = 0.$$

Let $\mathbf{y}(\xi)$ be a continuous function on the interval $[0, 1]$. It is easy to see that, as $\varepsilon \to 0+$, the vector function $\mathbf{y}_{(j)}(\xi,\varepsilon)$ tends to $\mathbf{y}(\xi)$ uniformly on $[0, 1]$, for any $j = 0,\ldots,s$.

The rest of the proof is almost an exact repetition of the proof of Theorem 1.1.2 (second step), in connection with the application to system (A.9) of the implicit function theorem [94]. From this alone we get a clear sense of condition (A.6): for its satisfaction $\mathbf{y}_{(j)}(\xi,\varepsilon)$ belongs to $\mathbf{C}[0, 1]$ only if the vector function $\mathbf{y}(\xi)$ belongs to $\mathbf{C}[0, 1]$, along with its first derivative, and $\varepsilon > 0$ is sufficiently small.

The only essential difference in the proof of the given theorem from the proof of Theorem 1.1.2 consists of the following. Consider the linear operator

$$\mathbf{T}_{(j)}(\varepsilon)\colon \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta}, \quad j = 1,\ldots,s,$$

acting on vector functions $\mathbf{z} \in \mathfrak{B}_{1,\Delta}$ according to the rule:

$$\mathbf{T}_{(j)}(\varepsilon)(\mathbf{z})(\xi) =$$
$$= \mathbf{z}_{(j)}(\xi,\varepsilon) = \left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\mathbf{G}}\mathbf{z}\left(\xi\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right).$$

It should be kept in mind that $\mathbf{T}_{(j)}(\varepsilon)$ can be considered on a much larger space, i.e. as a linear operator mapping the space $\mathfrak{B}_{0,\Delta}$ into itself. However, $\mathbf{T}_{(j)}(\varepsilon)$ is a continuous operator only as a transformation from $\mathfrak{B}_{1,\Delta}$ into $\mathfrak{B}_{0,\Delta}$. For example, the proof presented doesn't hold for a system of equations of neutral type where the deviating arguments enter into the expression with the higher derivatives (regarding classical systems of equations with deviating argument, see the monograph [47]).

Thus we prove the continuity of $\mathbf{T}_{(j)}(\varepsilon)$ in ε. This proof is based on the mean value theorem and the fact that the positive constant β can always be taken to be less than unity:

$$\begin{aligned}
&\left\|\mathbf{T}_{(j)}(\varepsilon_1) - \mathbf{T}_{(j)}(\varepsilon_2)\mathbf{z}\right\|_{0,\Delta} = \\
&= \left\|\left(1 + \gamma t_j (\varepsilon_1\xi)^{1/\beta}\right)^{-\mathbf{G}} \mathbf{z}\left(\xi\left(1 + \gamma t_j (\varepsilon_1\xi)^{1/\beta}\right)^{-\beta}\right) - \right.\\
&\left.- \left(1 + \gamma t_j (\varepsilon_2\xi)^{1/\beta}\right)^{-\mathbf{G}} \mathbf{z}\left(\xi\left(1 + \gamma t_j (\varepsilon_2\xi)^{1/\beta}\right)^{-\beta}\right)\right\|_{0,\Delta} \le \\
&\le \sup_{\xi\in[0,1],\varepsilon\in(0,\varepsilon_0)} \xi^{-\Delta+1/\beta}\gamma t_j \left\|-\mathbf{G}\left(1 + \gamma t_j (\varepsilon\xi)^{1/\beta}\right)^{-(\mathbf{G}+\mathbf{E})} \times\right.\\
&\times \mathbf{z}\left(\xi\left(1 + \gamma t_j (\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right) + \left(1 + \gamma t_j (\varepsilon\xi)^{1/\beta}\right)^{-(\mathbf{G}+(\beta+1)\mathbf{E})} \times\\
&\left.\times \left(1 + \gamma t_j (\varepsilon\xi)^{1/\beta} - \beta\xi\right)\mathbf{z}'\left(\xi\left(1 + \gamma t_j (\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right)\right\| \times\\
&\times \left|\varepsilon_1^{1/\beta} - \varepsilon_2^{1/\beta}\right| \le C\,\|\mathbf{z}\|_{1,\Delta}\left|\varepsilon_1^{1/\beta} - \varepsilon_2^{1/\beta}\right|,
\end{aligned}$$

where $\mathbf{z} \in \mathfrak{B}_{1,\Delta}$ is an arbitrary vector function and the constant $C > 0$ doesn't depend on $\mathbf{z}$.

The theorem is proved.

To illustrate the proof of the theorem, we consider a simple example.

Example A.1. We consider a system of two differential equations:

$$\dot{x}(t) = -\frac{1}{3}x^4(t), \quad \dot{y}(t) = -\frac{2}{3}x(t+t_1)y^2(t).$$

According to the terminology we have introduced, this system is quasihomogeneous with respect to the structure given by the matrix $\mathbf{G} = \operatorname{diag}(1/3, 2/3)$ which, as has already been noted, doesn't guarantee the existence of a particular solution in the form of a quasihomogeneous ray. It is, nonetheless, rather easy to "guess" at an asymptotic solution as $t \to \pm\infty$ of the system considered:

$$x(t) = t^{-1/3}, \quad y(t) = (t+t_1)^{-2/3}.$$

This asymptotic solution admits an expansion into a series of form (A.7)

$$x(t) = t^{-1/3}, \quad y(t) = t^{-2/3}\left(1 - \frac{2}{3}\frac{t_1}{t} + \frac{5}{9}\left(\frac{t_1}{t}\right)^2 - \dots\right).$$

The model system of ordinary differential equations

$$\dot{x} = -\frac{1}{3}x^4, \quad \dot{y}(t) = -\frac{2}{3}xy^2,$$

has a particular solution in the form of a ray,

$$x^{\pm}(t) = t^{-1/3}, \quad y^{\pm}(t) = t^{-2/3},$$

which represents the principal term of the asymptotic expansion of the particular solution we found for the full system.

Theorem A.1 has important applications in stability theory.

Theorem A.2. *Let the system of equations (A.1) of delay type* $(t_j < 0, j = 1, \ldots, s)$ *be semi-quasihomogeneous and let there exist a vector* $\mathbf{x}_0^- \in \mathbb{R}^n$, $\mathbf{x}_0^-$ *such that the equality*

$$\mathbf{G}\mathbf{x}_0^- = \mathbf{g}_q(\mathbf{x}_0^-).$$

holds. Then the trivial solution, $\mathbf{x}(t) \equiv \mathbf{0}$, *of (A.1) is unstable.*

Proof. This is immediate from the existence of a particular solution of the system (A.1): $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$.

Remark A.1. From Theorem A.1 it follows that, by fulfilling the specified hypothesis, a system of *advanced* type (i.e. $t_j > 0$, $j = 1, \ldots, s$) has a particular solution that is smooth on some *positive* half-line $[T, +\infty)$. This is a rather remarkable fact in that the Cauchy problem, as is known, isn't ÜÜwell posedÝÝ for the advanced problem.

The series constructed in Example A.1 are clearly convergent for $|t| > |t_1|$. In the general case, the problem of determining the convergence of the series (A.7) is more complicated than for ordinary differential equations and we won't dwell on it at length. Evidently the fact that a method of Chap. 1, based on the implicit function theorem, was applied to the system under investigation attests in favor of convergence. On the other hand, for the proof of the existence of a formal solution, we use Theorem 3.3.2, which is the route we expect to follow in analyzing *singular* problems for which formal series typically diverge. However, we remark that, as in the case of ordinary equations, the presence of a *nontrivial* linear part leads to divergence of the asymptotic series. The formulation of the theorem on the existence of asymptotic solutions in this case would require a discussion of the theory of the center manifold for systems of functional-differential equations, so that we will not present a general result here, but rather confine ourselves to considering examples analogous to Example 3.1.1. Some general results can be found in the article [64].

Example A.2. We consider a system of two equations

$$\dot{x} = -x(t) + y(t + t_1), \quad \dot{y} = -y^2(t), \quad t_1 < 0,$$

the second of which has the obvious solution $y(t) = t^{-1}$. Upon substitution into the first equation we obtain

$$x(t) = e^{-t} \int_{-\infty}^{t} (s + t_1)^{-1} e^s \, ds.$$

This function clearly belongs to the space $\mathbf{C}^{\infty}(-\infty, t_1)$, tends to zero as $t \to -\infty$ and its expansion into a formal series may diverge everywhere (although it is Borel summable [75]).

$$x(t) = \sum_{k=1}^{\infty} (k-1)!(t + t_1)^{-k}.$$

This series can be converted into a power series in whole negative powers of t. We find the asymptotic of its coefficients:

$$x(t) = e^{-t_1} \int_{-\infty}^{t_1} (\sigma + t)^{-1} e^{\sigma} \, d\sigma = -e^{-t_1} \sum_{k=1}^{\infty} \Gamma(k, -t_1) t^{-k},$$

where

$$\Gamma(k, \tau) = \int_{\tau}^{-\infty} u^{k-1} e^u \, du$$

is the incomplete Euler gamma function [91].

It is easy to see that the following bound holds:

$$\Gamma(k, \tau) \geq (k-1)! - k^{-1} \tau^k.$$

An account of the above method extends, generally speaking, to a broader class of systems. It evidently applies to functional-differential equations of the Volterra type in their most general form (for an account of the theory of these equations see the monograph [73]). Here we won't, however, introduce such generality and merely show how to apply this circle of ideas to autonomous functional-differential equations of the retarded type with discrete and distributed time lags. Thus we consider a system of equations to the form

$$\begin{aligned} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t - t_1), \ldots, \mathbf{x}(t - t_s), \textstyle\int_{-\infty}^{t} \mathbf{L}_1(t-u)\mathbf{x}(u)\, du, \ldots, \\ \ldots, \textstyle\int_{-\infty}^{t} \mathbf{L}_r(t-u)\mathbf{x}(u)\, du) \end{aligned} \tag{A.10}$$

where $t_1 > 0$, $i = 1, \ldots, s$ (for a system of retarded type with discrete time lags we will ascribe a *minus* sign), the $\mathbf{L}_j(\sigma)$, $j = 1, \ldots, r$, are matrix functions that are continuous on the positive interval $(0, +\infty)$, and

$$\mathbf{f} = \mathbf{f}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}\right)$$

is a smooth vector function in its arguments, where we have introduced the notation:

$$\mathbf{x}_{[1]} = \int_{-\infty}^{t} \mathbf{L}_1(t-u)\mathbf{x}(u)\,du, \ldots, \mathbf{x}_{[r]} = \int_{-\infty}^{t} \mathbf{L}_r(t-u)\mathbf{x}(u)\,du.$$

As before, let $\mathbf{x}(t) \equiv \mathbf{0}$ be the trivial solution of the system considered. Our goal will be to establish instability criteria of this solution in "supercritical" cases. If there are no terms with discrete lags $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$ on the right side of system (A.10), then (A.10) is a system of integro-differential equations of Volterra type (for the basic theory of these systems, see e.g. [35]). In noncritical cases an analogue of Lyapunov's first method has been worked out for systems of such form [162, 163].

We make additional assumptions regarding system (A.10). Usually we assume that the components of the matrices $\mathbf{L}_j(\sigma)$, $j = 1, \ldots, r$ are exponentially decreasing, i.e. that we have the inequalities

$$\|\mathbf{L}_j(\sigma)\| \le L_j e^{-a_j \sigma}, \quad \sigma \in (0, +\infty), \quad j = 1, \ldots, r.$$

We impose harsher conditions on the matrices $\mathbf{L}_j(\sigma)$, i.e. we suppose that they can be expanded in absolutely convergent series of the form:

$$\mathbf{L}_j(\sigma) = \sum_{l=1}^{\infty} e^{-a_{jl}\sigma} \left(\mathbf{M}_{jl}\cos(b_{jl}\sigma) + \mathbf{N}_{jl}\sin(b_{jl}\sigma)\right). \tag{A.11}$$

We introduce the notations:

$$\begin{aligned}
\mathbf{M}_{jl}^{(0)} &= \mathbf{M}_{jl}, \quad \mathbf{N}_{jl}^{(0)} = \mathbf{N}_{jl}, \\
\mathbf{M}_{jl}^{(p+1)} &= \frac{1}{a_{jl}^2 + b_{jl}^2}\left(a_{jl}\mathbf{M}_{jl}^{(p)} + b_{jl}\mathbf{N}_{jl}^{(p)}\right), \\
\mathbf{N}_{jl}^{(p+1)} &= \frac{1}{a_{jl}^2 + b_{jl}^2}\left(b_{jl}\mathbf{M}_{jl}^{(p)} - a_{jl}\mathbf{N}_{jl}^{(p)}\right),
\end{aligned}$$

for $p = 0, 1, \ldots$.

The following assumption, stronger than the assumption of absolute convergence for the series (A.11), consists of the matrix series

$$\sum_{l=1}^{\infty} \mathbf{M}_{jl}^{(p)}, \quad \sum_{l=1}^{\infty} \mathbf{N}_{jl}^{(p)}, \tag{A.12}$$

which is absolutely convergent to certain matrices $\mathbf{M}_j^{(p)}, \mathbf{N}_j^{(p)}$, $j = 1, \ldots, r$, $p = 0, 1, \ldots$

For a system of type (A.10) we introduce concepts of quasihomogeneity and semi-quasihomogeneity.

Definition A.3. We say that the system of equations (A.10) is *quasihomogeneous* with respect to the quasihomogeneous structure generated by the matrix $\mathbf{G}$ and will denote its right side by $\mathbf{f} = \mathbf{f}_{\mathrm{q}}$ if, for any system $\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}$ and any $\mu \in \mathbb{R}^{+}$, we have the equality

$$\begin{aligned} &\mathbf{f}_{\mathrm{q}}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}, \mu_{[1]}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{[r]}^{\mathbf{G}}\mathbf{x}\right) = \\ &= \mu^{\mathbf{G}+\mathbf{E}}\mathbf{f}_{\mathrm{q}}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}\right). \end{aligned} \tag{A.13}$$

Definition A.4. We say that the system of equations (A.10) is *semi-quasihomogeneous* with respect to the structure generated by the matrix $\mathbf{G}$ if its right side is represented as a formal sum

$$\mathbf{f}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right)$$

such that, for any $\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}$ and $\mu \in \mathbb{R}^{+}$ for any m–th form $\mathbf{f}_{\mathrm{q}+m}$, we have the equality

$$\begin{aligned} &\mathbf{f}_{\mathrm{q}+m}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}, \mu_{[1]}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{[r]}^{\mathbf{G}}\mathbf{x}\right) = \\ &= \mu^{\mathbf{G}+m\beta\mathbf{E}}\mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}\right), \end{aligned} \tag{A.14}$$

where β is some positive number.

We consider a model system of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathbf{g}_{\mathrm{q}}(\mathbf{x}), \tag{A.15}$$

where

$$\mathbf{g}_{\mathrm{q}}(\mathbf{x}) = \mathbf{f}_{\mathrm{q}}\left(\mathbf{x}, \ldots, \mathbf{x}, \mathbf{M}_1^{(1)}\mathbf{x}, \ldots, \mathbf{M}_r^{(1)}\mathbf{x}\right).$$

We will determine a connection between instability for the model system (A.15) and instability for the full system (A.10). However, instability for the full system bears a *formal* character that will be elaborated below.

Theorem A.3. *Let the system of equations (A.10) be semi-quasihomogeneous and suppose there exists a nonzero vector* $\mathbf{x}_0^-$ *such that we have the following equality:*

$$\mathbf{G}\mathbf{x}_0^- = g_{\mathrm{q}}(\mathbf{x}_0^-).$$

Then system (A.10) has a particular formal solution represented in the form of a series, each term of which converges to zero as $t \to -\infty$.

If the series referred to in the statement of Theorem A.3 converges, then system (A.10) will have a particular solution

$$\mathbf{x}(t) \to \mathbf{0} \ \text{as} \ t \to -\infty,$$

which points to instability. We show below, in a concrete example, that this series can diverge. In the case considered, however, the theory of Kuznetsov [125, 126] is inapplicable, since this was developed only for systems of *ordinary* differential equations. We therefore speak only of *formal* instability.

Proof of Theorem A.3. The required particular formal solution will be sought in the usual form:

$$\mathbf{x}_{[j]}(t) = (-t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}\,(\ln(-t))\,(-t)^{-k\beta}. \tag{A.16}$$

Here we can again use Theorem 3.3.2. Applying integration by parts an infinite number of times to the vector functions $\mathbf{x}(t), \ldots, \mathbf{x}_{[r]}(t)$, we obtain the formal expansion

$$\mathbf{x}_{[j]}(t) = \sum_{p=0}^{\infty} (-1)^p \mathbf{M}_j^{(p+1)} \mathbf{x}^{(p)}(t)$$

for each $j = 1, \ldots, r$. Here we use the absolute convergence of the series (A.12).

Such a representation of the quantities $\mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}$ again converts the system being investigated into a system of ordinary differential equations of type (A.8). From the equalities (A.13) and (A.14), it follows that this system (implicit with respect to higher derivatives!) is semi-quasihomogeneous with respect to the structure given by the matrix $\mathbf{G}$ (see Definition 3.3.5), where its quasihomogeneous truncation coincides with system (A.15) which has, by the hypothesis of the theorem, a particular solution in the form of a ray tending to the point $\mathbf{x}= \mathbf{0}$ as $t \to -\infty$. Application of Theorem 3.3.2 allows us to assert that system (A.10) has a formal particular solution in the form of series (A.16).

The theorem is proved.

Example A.3. We consider the system of equations:

$$\dot{x} = x^2(t) \left(1 + \int_{-\infty}^{t} e^{-(t-u)} y(u)\, du \right), \quad \dot{y}(t) = y^2(t).$$

According to the definitions introduced above, this system is semi-(quasi)-homogeneous with respect to the structure generated by the identity matrix $\mathbf{E}$. The corresponding model system

$$\dot{x} = x^2, \quad \dot{y} = y^2$$

has the obvious asymptotic solution $x^-(t) = y^-(t) = -\dfrac{1}{t}$.

The second equation of the original system is easily integrated. Its asymptotic solution has the simple form:

$$y(t) = -t^{-1}.$$

But then the result of an integral transformation of the function $y(t)$,

$$\int_{-\infty}^{t} e^{-(t-u)} y(u)\, du = \sum_{k=1}^{\infty} (k-1)! t^{-k},$$

is represented as an everywhere divergent series.

We denote by $\phi(t)$ the formal expansion of the following integral:

$$\phi(t) = \int_{-\infty}^{t} \left(\int_{-\infty}^{u} e^{-(u-v)} v^{-1}\, dv - u^{-1} \right) du = -\sum_{k=2}^{\infty} (k-2)! t^{-k+1}.$$

Now we can find an expansion of a formal asymptotic solution for the first equation of the system considered:

$$x(t) = -t^{-1} \left(1 + t^{-1} \ln(-t) + t^{-1} \phi(t)\right)^{-1} =$$
$$= (-t)^{-1} \left(1 - (-t)^{-1} \ln(-t) + \sum_{k=2}^{\infty} (-1)^{k-1} (k-2)! (-t)^{-k} \right)^{-1}.$$

Having converted the last expression to an expansion in powers of $-t$, we obtain a series of type (A.16) which will, of course, diverge. As is to be expected, the principal term of this expansion will coincide with $x^{-}(t)$.

We consider some applications of the theory we have constructed.

Example A.4. In (1.63) of the first chapter we considered a so-called logistical system of ordinary differential equations, (1.64), describing processes of mutual interaction. We should, however, keep in mind that the impact upon the numbers of individuals in populations by the birthrates of species proceeds with a certain time lag, as reflected in the following mathematical model, described by a system of functional differential equations containing discrete and continuous time lags [180, 189]:

$$\dot{N}^i(t) =$$
$$= N^i(t) \left(k_i + b_i^{-1} \sum_{p=1}^{n} \left(a_p^i N^p(t - t_{ip}) + \int_{-\infty}^{t} f_p^i(t-u) N^p(u)\, du \right) \right), \tag{A.17}$$

where $i = 1, \ldots, n$.

Here the $t_{ip} > 0$, $i, p = 1, \ldots, n$ are discrete lag arguments, and the matrix

$$\mathbf{F}(\sigma) = \left(f_p^i(\sigma)\right)_{i,p=1}^n$$

possesses the very same properties as the matrices $\mathbf{L}_1(\sigma), \ldots, \mathbf{L}_r(\sigma)$ that figure in the statement of Theorem A.3

We likewise introduce the following notation:

$$\tilde{\mathbf{F}} = \left(\tilde{f}_p^i\right)_{i,p=1}^n = \left(\int_0^\infty f_p^i(\sigma)\, d\sigma\right)_{i,p=1}^n .$$

We will find sufficient conditions for the instability of the trivial solution of system (A.17) for the "supercritical" case, where the values of all the "Malthusian birthrates" k_i, $i = 1, \ldots, n$, i.e. the average number of individuals in populations when left to themselves, equal zero. Thus the object of our investigations will be the system

$$\begin{gathered}\dot{N}^i(t) = \\ = b_i^{-1} N^i(t) \sum_{p=1}^n \left(a_p^i N^p(t - t_{ip}) + \int_{-\infty}^t f_p^i(t-u) N^p(u)\, du\right), \\ i = 1, \ldots, n.\end{gathered} \tag{A.18}$$

This system (A.18) of functional-differential equations of retarded type is (quasi)homogeneous, according to the definitions introduced. The corresponding model system of ordinary differential equations has the following appearance:

$$\dot{N}^i = b_i^{-1} N^i \sum_{p=1}^n \left(a_p^i + \tilde{f}_p^i\right) N^p, \quad i = 1, \ldots, n. \tag{A.19}$$

System (A.19) has an increasing solution in the form of a linear ray,

$$\mathbf{N}^-(t) = (-t)_0^{-1} \mathbf{N}_0^-, \quad \mathbf{N} = (N^1, \ldots, N^n),$$

provided the system, when solved with respect to $\mathbf{N}_0^- = (N_0^{-1}, \ldots, N_0^{-n})$, is a linear system of algebraic equations

$$\sum_{p=1}^n \left(a_p^i + \tilde{f}_p^i\right) N_0^{-p} = b_i .$$

Here, of course, we need only look for the *positive* solution of this linear system, since the population sizes must be positive.

The asymptotic particular solution as $t \to -\infty$ of the truncated system (A.19) generates a formal solution of the full system (A.18) of the form

$$\mathbf{N}(t) = \sum_{k=1}^{\infty} \mathbf{N}_k \left(\ln(-t)\right) (-t)^k .$$

In the case of convergence of this series, the existence of a similar solution would indicate instability. If a continuous time lag is absent, then by Theorem A.2 we will have Lyapunov instability. This instability has an explosive character: populations which find themselves on the verge of extinction at some initial moment begin to grow by a roughly exponential law, whereby the number of individuals in them theoretically can become infinite in finite time. This interesting effect, where populations "pull themselves up by their hair" (like baron Münchhausen) from the "morass of extinction", was discussed previously in the article [64] for systems with discrete time lags.

Appendix B
Arithmetic Properties of the Eigenvalues of the Kovalevsky Matrix and Conditions for the Nonintegrability of Semi-quasihomogeneous Systems of Ordinary Differential Equations

In the introduction we mentioned, in passing, a series of papers dedicated to problems of integrability of equations of motion and to the chaotic behavior of trajectories of nonlinear dynamical systems from the point of view of bifurcation, in the complex time plane, of particular solutions with generalized power asymptotic. It was noted that there are no rigorous results that allow us to treat properties such as the integrability of these solutions. In this appendix we state a series of results [65] concerning the nonintegrability of semi-quasihomogeneous differential equations obtained by using the proven existence of solutions with generalized power asymptotic. This should be regarded as a zero-th order theory, since it only contains properties of quasihomogeneous truncations and principal terms of asymptotic expansions of solutions. The results obtained constitute an extension of those in a paper by H. Yoshida [197], where it was first noted that the eigenvalues of the Kovalevsky matrix of an integrable quasihomogeneous system of equations must satisfy certain resonance relationships.

Here we introduce a different treatment of this idea, based on different considerations. In order to better understand the essence of this phenomenon, we consider the following simplified situation. Let there be given a system of differential equations with analytic right side:

$$\dot{\mathbf{u}} = \mathbf{h}(\mathbf{u}), \quad \mathbf{u} \in \mathbb{C}^{\mathbf{n}}, \tag{B.1}$$

for which the origin $\mathbf{u} = \mathbf{0}$ is a critical point ($\mathbf{h}(\mathbf{0}) = \mathbf{0}$).

Let $\mathbf{A} = \mathbf{dh}(\mathbf{0})$ be the Jacobian matrix of the vector field $\mathbf{h}(\mathbf{u})$, computed at the critical point. For simplicity, we further assume that the matrix $\mathbf{A}$ is diagonalizable and that the coordinates $(u^1, \dots, u^n)$ are chosen so that $\mathbf{A}$ already has diagonal form: $\mathbf{A} = \operatorname{diag}(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_\mathbf{n})$.

The basic idea amounts to the observation that, if system (B.1) has an analytic integral, then the eigenvalues of the matrix $\mathbf{A}$ must satisfy certain resonance conditions. More precisely, we have

Lemma B.1. *Suppose* $\det \mathbf{A} \neq \mathbf{0}$ *and that, for any choice of nonnegative integers* $k_1, \dots, k_n$, $k_j \in \mathbb{N} \cup \{0\}$, *where* $\sum_{j=1}^{n} k_j \geq 1$,

V.V. Kozlov and S.D. Furta, *Asymptotic Solutions of Strongly Nonlinear Systems of Differential Equations*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-642-33817-5,

$$\sum_{j=1}^{n} k_j \lambda_j \neq 0. \tag{B.2}$$

Then the system (B.1) has no integral that is infinitely differentiable in the neighborhood of $\mathbf{u} = \mathbf{0}$*, i.e. which admits a nontrivial formal Maclaurin expansion.*

Proof. Suppose there exists a smooth function $\phi(\mathbf{u})$ that admits expansion into a nontrivial formal Maclaurin series and is an integral of system (B.1). This function must satisfy the following first order partial differential equation:

$$\langle d\phi(\mathbf{u}), \mathbf{h}(\mathbf{u})\rangle = 0, \tag{B.3}$$

where the symbol $\langle\cdot,\cdot\rangle$ now denotes the Hermite scalar product on $\mathbb{C}^n$.

Without loss of generality we may suppose that $\phi(\mathbf{0}) = 0$. Consider the expansion of $\phi(\mathbf{u})$ into a series in the neighborhood of $\mathbf{u} = \mathbf{0}$:

$$\phi(\mathbf{u}) = \sum_{\mathbf{s}=\mathbf{1}}^{\infty} \phi_{(\mathrm{s})}(\mathbf{u}),$$

where the $\phi_{(s)}(\mathbf{u})$, $s = 1, 2, \dots,$ are homogeneous polynomials in $\mathbf{u}$ of degree s.

We consider first the form of the expansion of the integral $\phi(\mathbf{u})$:

$$\phi_{(1)}(\mathbf{u}) = \langle \mathbf{b}, \mathbf{u}\rangle,$$

where $\mathbf{b} \in \mathbb{C}^{\mathbf{n}}$ is a constant vector. If we equate the terms of (B.3) that are linear in $\mathbf{u}$, we obtain the equality

$$\langle \mathbf{b}, \mathbf{A}\mathbf{u}\rangle = \mathbf{0}, \tag{B.4}$$

which must be satisfied for any $\mathbf{u} \in \mathbb{C}^{\mathbf{n}}$.

Then it follows from (B.4) that the vector $\mathbf{b}$ is an eigenvector of the matrix $\mathbf{A}^*$ (the symbol $(\ )^*$ denotes Hermite conjugation) with zero eigenvalue, which contradicts the condition $\det \mathbf{A} \neq \mathbf{0}$. Therefore $\mathbf{b} = \mathbf{0}$.

Suppose now that we have proved that $\phi_{(1)} \equiv \ldots \equiv \phi_{(s-1)} \equiv 0$. Then from (B.3) it follows that

$$\left\langle d\phi_{(s)}(\mathbf{u}), \mathbf{A}\mathbf{u}\right\rangle = 0. \tag{B.5}$$

We denote by $D^{(s)}\phi(\mathbf{v}, \mathbf{u})$ the s–th differential of the function ϕ at the point $\mathbf{v}$, computed at the vector $\mathbf{u}$.

On the basis of (B.5) we can make an important observation: the first nontrivial form in the expansion of the integral of system (B.1),

$$\phi_{(s)}(\mathbf{u}) = \mathbf{D}^{(\mathbf{s})}\phi(\mathbf{0}, \mathbf{u}),$$

is an integral of the linear system

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}.$$

Equation (B.5) can be rewritten in the following form:

$$\sum_{j=1}^{n} \lambda_j \frac{\partial \phi_{(s)}}{\partial u^j}(\mathbf{u}) u^j = 0. \tag{B.6}$$

We rewrite the homogeneous polynomials $\phi_{(s)}(\mathbf{u})$ in the form of a sum of elementary monomials:

$$\phi_{(s)}(u^1, \ldots, u^n) = \sum_{k_1+\ldots+k_n=s} \phi_{k_1\ldots k_n} (u^1)^{k_1} \ldots (u^n)^{k_n}.$$

Then (B.6) assumes the form:

$$\sum_{k_1+\ldots+k_n=s} (k_1\lambda_1 + \ldots + k_n\lambda_n)\, \phi_{k_1\ldots k_n} (u^1)^{k_1} \ldots (u^n)^{k_n} = 0, \tag{B.7}$$

and it follows from equality (B.7) that, if only the coefficient $\phi_{k_1\ldots k_n} \neq$ is nonzero, then

$$k_1\lambda_1 + \ldots + k_n\lambda_n = 0,$$

which contradicts condition (B.2).

The lemma is proved.

Remark B.1. The explicit requirement that $\det \mathbf{A} \neq \mathbf{0}$ can in fact be dropped, since condition (B.2) on the absence of resonances itself contains this requirement.

We now consider a more complicated situation. Let there be given a semi-quasihomogeneous system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}.) \tag{B.8}$$

Here it will be more convenient for us to use the traditional definitions of quasihomogeneity and semi-quasihomogeneity 1.1.2 and 1.1.4, whose introduction was based on the Newton manifold technique. Then the elements of the diagonal matrix $\mathbf{G} = \alpha\mathbf{S}$, $\alpha = \dfrac{1}{q-1}$, $\mathbf{S} = \mathrm{diag}\,(\mathbf{s_1}, \ldots, \mathbf{s_n})$, $s_j \in \mathbb{N} \cup \{0\}$, $j = 1, \ldots, n$, $q \in \mathbb{N}$, $q \neq 1$, that determine the quasihomogeneous structure, are rational and nonnegative.

We will consider two different problems:

(a) If system (B.8) is positive semi-quasihomogeneous, we investigate for the system the question of existence of smooth integrals, represented in the form of nontrivial Maclaurin series

$$F(x) = F(x^1, \ldots, x^n) = \sum_{k_1\geq 0,\ldots,k_n\geq 0}^{\infty} F_{k_1\ldots k_n} (x^1)^{k_1} \ldots (x^n)^{k_n}; \tag{B.9}$$

(b) If system (B.8) is negative semi-quasihomogeneous, then we will look for polynomial integrals, i.e. integrals for which expansion (B.9) contains only a finite number of terms.

We have:

Lemma B.2. *If the semi-quasihomogeneous system (B.8) has a nontrivial integral of the form (B.9), polynomial in the case of positive semi-quasihomogeneity, then the corresponding truncation of the system has a quasihomogeneous integral.*

Proof. We consider first case (a). The function $F(\mathbf{x})$ can be assumed to be positive semi-quasihomogeneous and re-expanded in a series of quasihomogeneous form, conforming to the quasihomogeneous structure with matrix $\mathbf{S}$:

$$F(\mathbf{x}) = \sum_{m=0}^{\infty} F_{N+m}(\mathbf{x}).$$

After the substitution $\mathbf{x} \mapsto \lambda^{\mathbf{S}}\mathbf{x}$ this function takes the form

$$F(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^N \sum_{m=0} \lambda^m F_{N+m}(\mathbf{x}).$$

Regardless of the sign of semi-quasihomogeneity the system of equations (B.8), after the substitution $\mathbf{x} \mapsto \mu^{\mathbf{G}}\mathbf{x}$, $t \mapsto \mu^{-1}t$, is written in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mu) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}) + \sum_{m=1} \mu^{\beta m}\mathbf{f}_{\mathrm{q}+\chi\mathbf{m}}(\mathbf{x}). \tag{B.10}$$

Here $\mu = \lambda^{\mathrm{q}-1}$, $\beta = \pm\alpha$, $\chi = \operatorname{sign}\beta$.
In case (a), as $\mu \to 0$, Eq. (B.10) is transformed into the truncated system

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}). \tag{B.11}$$

System (B.10) clearly has, for any $\mu \in (0, +\infty)$, the integral

$$F(\mathbf{x},\mu) = F_N(\mathbf{x}) + \sum_{m=1} \mu^{\beta m} F_{N+\chi m}(\mathbf{x}), \tag{B.12}$$

(in case (a), $\chi = +1$) as $\mu \to 0+$, passing to the quasihomogeneous function

$$F_N(\mathbf{x}), \tag{B.13}$$

which in its turn is an integral of the truncated system (B.11).

Case (b) is treated analogously. The polynomial integral $F(\mathbf{x})$ decomposes into a *finite* sum of quasihomogeneous form, generated by the structure with matrix $\mathbf{G}$. Therefore this function may be taken to be negative semi-quasihomogeneous, whence the relation

$$F(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^N \sum_{m=0}^{M} \lambda^{-m} F_{N-m}(\mathbf{x}), \quad N \geq M.$$

In case (b), system (B.10) also has an integral of type (B.12), whose expansion in powers of μ^β, $\beta = -\alpha$ in fact contains only a finite number of terms. When $\mu \to +\infty$, system (B.10) passes to the truncated system (B.11), and the integral (B.12) to the integral (B.13) of the truncated system.

The lemma is proved.

We now consider more closely the truncated system (B.11), which we suppose has a particular solution in the form of a (possibly complex) quasihomogeneous ray

$$\mathbf{x}_{(0)}(t) = t^{-\mathbf{G}}\mathbf{x_0}, \tag{B.14}$$

where $\mathbf{x_0} \in \mathbb{C}^{\mathbf{n}}$ is a constant nonzero vector satisfying the equation

$$\mathbf{G}\mathbf{x_0} + \mathbf{f}_{\mathrm{q}}(\mathbf{x_0}) = 0.$$

It is clear that, if $F_N(\mathbf{x})$ is an integral of system (B.11), then $F_N(\mathbf{x_0}) = 0$.

We compute the Kovalevsky matrix for the linear system

$$\mathbf{K} = \mathbf{G} + d\mathbf{f}_{\mathrm{q}}(\mathbf{x_0}).$$

Let $\rho_1, \ldots, \rho_n$ be the roots of the characteristic equation

$$\det(\mathbf{K} - \rho\mathbf{E}) = \mathbf{0}.$$

For simplicity we will hereafter always assume that the Kovalevsky matrix reduces to diagonal form.

We have:

Lemma B.3. *Suppose that, for any choice of the numbers* $(k_1 \ldots k_n)$, $k_j \in \mathbb{N} \cup \{0\}$, $\sum_{j=1}^n k_j \geq 1$, *the inequality*

$$\sum_{j=1}^{n} k_j \rho_j \neq 0. \tag{B.15}$$

is satisfied. Then any quasihomogeneous integral of the truncated system (B.11) is trivial.

Proof. Suppose system (B.11) has a nontrivial quasihomogeneous integral (B.13). After the change of variables

$$\mathbf{x} = t^{-\mathbf{G}}(\mathbf{x_0} + \mathbf{u})$$

the system of equations (B.11) is transformed to the system

$$t\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \hat{\mathbf{f}}(\mathbf{u}), \tag{B.16}$$

where $\mathbf{K}$ is the Kovalevsky matrix and $\hat{\mathbf{f}}(\mathbf{u})$ (with $\hat{\mathbf{f}}(\mathbf{0}) = \mathbf{0}$, $d\hat{\mathbf{f}}(\mathbf{0}) = \mathbf{0}$) is the vector field computed by the formula

$$\hat{\mathbf{f}}(\mathbf{u}) = \mathbf{f}_{\mathrm{q}}(\mathbf{x_0} + \mathbf{u}) - \mathbf{f}_{\mathrm{q}}(\mathbf{x_0}) - d\,\mathbf{f}_{\mathrm{q}}(\mathbf{x_0})\mathbf{u}.$$

If we make a logarithmic change of time $\tau = \ln t$, then system (B.16) transforms into the system

$$\mathbf{u}' = \mathbf{K}\mathbf{u} + \hat{\mathbf{f}}(\mathbf{u}), \tag{B.17}$$

where the prime signifies differentiation with respect to "logarithmic time" τ of the analogous system (B.1) that was considered at the beginning of this appendix.

However, the originally *autonomous* integral (B.13) of system (B.11) is transformed into a *time-dependent* integral of system (B.17):

$$F_N(\mathbf{x}) = e^{-\alpha N \tau} F_N(\mathbf{x_0} + \mathbf{u}).$$

We make an additional substitution $u^0 = e^{-\alpha\tau}$ and consider the extended system of equations

$$u^{0\prime} = -\alpha u^0, \quad \mathbf{u}' = \mathbf{K}\mathbf{u} + \hat{\mathbf{f}}(\mathbf{u}). \tag{B.18}$$

If the quasihomogeneous system (B.11) has a nontrivial quasihomogeneous integral (B.13), then the autonomous extended system (B.18) has a nontrivial *autonomous* integral

$$\phi(u^0, \mathbf{u}) = (u^0)^N F_N(\mathbf{x_0} + \mathbf{u}).$$

Then, by virtue of Lemma B.1, there exists a set of nonnegative integers $(k_0^*, k_1^*, \ldots, k_n^*)$, $\sum_{j=0}^{n} k_j^* \geq 1$, such that

$$-\alpha N k_0^* + \sum_{j=1}^{n} k_j^* \rho_j = 0. \tag{B.19}$$

According to Lemma 1.1.2, $\rho = -1$ is always an eigenvalue of the Kovalevsky matrix $\mathbf{K}$, so that without loss of generality we may assume that the first eigenvalue ρ_1 of this matrix equals -1. We rewrite the equality (B.19) in the form

$$-1(N k_0^* + (q-1)k_1^*) + (q-1)\sum_{j=2}^{n} k_j^* \rho_j = 0.$$

However, according to the hypothesis of the lemma (cf. (B.15)), this equation be cannot be satisfied.

The lemma is proved.

In proving Lemma B.3 based on Lemma B.1, we implicitly used a fact that we will be need in the sequel.

Remark B.2. Suppose the truncation of system (B.8) has a nontrivial quasihomogeneous integral $F_N(\mathbf{x})$. Then the linear system

$$\mathbf{u^0}' = -\alpha \mathbf{u^0}, \quad \mathbf{u}' = \mathbf{K}\mathbf{u} \tag{B.20}$$

has a homogeneous integral of the form

$$\phi_{(s+N)}(u^0, \mathbf{u}) = (u^0)^N D^{(s)} F_N(\mathbf{x_0}, \mathbf{u}),$$

where $s \geq 1$ is a natural number such that

$$D^{(p)} F_N(\mathbf{x_0}, \mathbf{u}) \equiv 0, \quad p = 1, \ldots, s, \quad D^{(s)} F_N(\mathbf{x_0}, \mathbf{u}) \not\equiv 0.$$

The proof of the following theorem on nonintegrability consists of a succession of applications of Lemmas B.2 and B.3.

Theorem B.1. *Suppose that system (B.8) has a infinitely differentiable right side and is semi-quasihomogeneous with respect to some structure given by the diagonal matrix* $\mathbf{G}$ *with rational nonnegative elements. Let* $\rho_1, \ldots, \rho_n$ *be the eigenvalues (possibly complex) of the diagonalizable Kovalevsky matrix, computed for one of the solutions of the truncated system (B.11). Suppose, for any choice of numbers* $(k_1, \ldots, k_n)$, $k_j \in \mathbb{N} \cup \{0\}$, $\sum_{j=1}^{n} k_j \geq 1$, *that we have the inequality (B.15). Then system (B.8) has no semi-quasihomogeneous integral. If system (B.8) is positive semi-quasihomogeneous, then (B.8) likewise doesn't have any infinitely differentiable integral that is represented as a nontrivial formal Maclaurin series (B.9).*

A much more complicated and interesting integrability problem arises in the case where system (B.8) has a nontrivial integral. The question usually arises in the following way: do their exist other integrals in the given class, functionally independent from those that are known?

We first present the following result, stated and proved in the paper [197].

Lemma B.4. *We suppose that the quasihomogeneous system (B.11) has a quasihomogeneous integral (B.13) that is nondegenerate for any particular solution of this system of quasihomogeneous ray type* $\mathbf{x_{(0)}}(\mathbf{t}) = \mathbf{t}^{-\mathbf{G}}\mathbf{x_0}$, *i.e.* $dF_N(\mathbf{x_0}) \neq \mathbf{0}$. *Let* $Q = \alpha N$ *be a rational quasihomogeneous "degree" of this integral:* $F_N(\mu^{\mathbf{G}}\mathbf{x}) = \mu^Q F_N(\mathbf{x})$. *Then* Q *is one of the eigenvalues of the Kovalevsky matrix.*

Proof. We introduce the notation:

$$\mathbf{b} = dF_N(\mathbf{x_0}).$$

Then the auxiliary system (B.18) has an integral, which can be expanded in a series of homogeneous forms in the variables $(u^0, \mathbf{u})$:

$$\phi(u^0, \mathbf{u}) = (u^0)^N F_N(\mathbf{x_0} + \mathbf{u}) = \sum_{s=1}^{\infty} \phi_{(s)}(u^0, \mathbf{u}),$$

where

$$\phi_{(1)}(u^0, \mathbf{u}) \equiv \ldots \equiv \phi_{(N)}(u^0, \mathbf{u}) \equiv 0, \quad \phi_{(N+1)}(\mathbf{u^0}, \mathbf{u}) \equiv (u^0)^N \langle \mathbf{b}, \mathbf{u} \rangle.$$

According to Remark B.2, the form $\phi_{(N+1)}(u^0, \mathbf{u})$ is an integral of the linear system (B.20), from which it immediately follows that, for any $\mathbf{u} \in \mathbb{C}^{\mathbf{n}}$,

$$\langle \mathbf{b}, (\mathbf{K} - \alpha N \mathbf{E})\mathbf{u} \rangle = 0.$$

Since the matrix $\mathbf{K}$ is diagonalizable by assumption, it follows from the last relation that the vector $\mathbf{b}$ is its eigenvector with eigenvalue Q.

The lemma is proved.

Above we stipulated that $\rho_1 = -1$ is the *first* eigenvalue of the Kovalevsky matrix, with eigenvector $\mathbf{p} = \mathbf{f}_{\mathrm{q}}(\mathbf{x_0})$. Without loss of generality we may suppose that the vector $\mathbf{p}$ is parallel to the *first* basis vector $\mathbf{e_1} = (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0})$. Using Lemma B.4, we may assume that $\rho_N = Q$ is the *last* eigenvalue of the Kovalevsky matrix and that $\mathbf{b}$ is parallel to the *last* basis vector $\mathbf{e_n} = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{1})$.

Thus one of the resonance relationships satisfied by the eigenvalues of the Kovalevsky matrix has, in this case, the form

$$-k_1 + Qk_n = 0, \quad k_1 = r, \quad k_n = p,$$

where $Q = p/r$, $\gcd(p, r) = 1$.

We have the following result on the nonintegrability of a system of type (B.8), based on the arithmetic properties of the Kovalevsky matrix.

Theorem B.2. *Suppose that the system (B.8), with infinitely differentiable right side, is semi-quasihomogeneous with respect to some structure with diagonal matrix* $\mathbf{G}$ *with nonnegative rational elements, and that it has a nontrivial integral* $F(\mathbf{x})$ *that is polynomial in the case of a negative semi-quasihomogeneous system, or infinitely differentiable and represented as a formal nontrivial Maclaurin series in the case of a positive semi-quasihomogeneous system. Let the quasihomogeneous truncation of this integral* $F_N(\mathbf{x})$ *be nondegenerate for one of the solutions of the truncated system (B.11) in the form of a ray. Suppose likewise that the Kovalevsky matrix* $\mathbf{K}$*, computed* *at this solution, is diagonalizable and that its first* $n-1$ *eigenvalues* $\rho_1, \dots, \rho_{n-1}$ *are such that for any choice of (possibly negative) integers* $k_1, \dots, k_{n-1}$, $k_j \in \mathbb{Z}$, $\sum_{j=1}^{n-1} |k_j| \neq 0$, *the inequality*

$$\sum_{j=1}^{n-1} k_j \rho_j \neq 0. \tag{B.21}$$

is satisfied. Then any other integral $G(\mathbf{x})$ *of system (B.8), polynomial in the negative semi-quasihomogeneous case and represented in the form of a formal power series in the positive semi-quasihomogeneous case, is functionally dependent on* $F(\mathbf{x})$*, i.e. there exists a smooth function* $H: \mathbb{R} \to \mathbb{R}$ *such that*

$$G(\mathbf{x}) = H(F(\mathbf{x})). \tag{B.22}$$

The proof depends on several auxiliary assertions.

Lemma B.5. *Suppose that system (B.8) has a smooth integral* $F(\mathbf{x})$ *and that any nontrivial quasihomogeneous integral* $G_K(\mathbf{x})$ $(G_K(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^K G_K(\mathbf{x}))$ *of the truncated system (B.11) is a function of the truncation* $F_N(\mathbf{x})$ *of the integral* $F(\mathbf{x})$*, i.e.*

$$G_K(\mathbf{x}) = \Phi\left(\mathbf{F_N}(\mathbf{x})\right) \tag{B.23}$$

(The function Φ: $R \to R$ generally might be just an elementary monomial.) Then any nontrivial integral $G(x)$ of system (B.8) is a smooth function of $F(x)$ (see (B.22)).

Proof. Making the change of variables $\varepsilon = \mu^\beta$ in (B.10), we obtain

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\varepsilon}) = \sum_{\mathbf{m=0}}^{\infty} \boldsymbol{\varepsilon}^{\mathbf{m}} \mathbf{f}_{\mathrm{q}+\chi\mathbf{m}}(\mathbf{x}). \tag{B.24}$$

As $\varepsilon \to 0$, system (B.24) is transformed into the truncated system (B.11). System (B.24) has the integral form

$$F(\mathbf{x},\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m \mathbf{F}_{\mathbf{N}+\chi\mathbf{m}}(\mathbf{x}), \tag{B.25}$$

which, as $\varepsilon \to 0$, is transformed into an integral of the truncated system (B.11).

If (B.8) has an additional integral $G(\mathbf{x})$, then (B.24) likewise has an additional integral of type analogous to (B.25):

$$G(\mathbf{x},\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m G_{\mathbf{K}+\chi\mathbf{m}}(\mathbf{x}), \tag{B.26}$$

As $\varepsilon \to 0$, the integral (B.26) passes to the truncation of $G_K(\mathbf{x})$, which is an integral of system (B.11).

The function in the variables $\mathbf{x}, \boldsymbol{\varepsilon}$,

$$G^{(1)}(\mathbf{x},\varepsilon) = G^{(0)}(\mathbf{x},\varepsilon) - \Phi^{(0)}\left(F(\mathbf{x},\varepsilon)\right),$$

where we have introduced the notations $G^{(0)}(\mathbf{x},\varepsilon) = G(\mathbf{x},\varepsilon)$, $\Phi^{(0)}(F) = \Phi(F)$, is likewise an integral of (B.24) and, by (B.23) and has a first order of smallness in ε.

We consider the following function:

$$G^{(1)}_{K_1}(\mathbf{x}) = \lim_{\varepsilon\to 0} \varepsilon^{-1} G^{(1)}(\mathbf{x},\varepsilon).$$

Since the integrals (B.25) and (B.26) are power series, whose coefficients are quasihomogeneous functions, $G^{(1)}_{K_1}(\mathbf{x})$ is likewise a quasihomogeneous function of some degree K_1, where $G^{(1)}_{K_1}(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^{K_1} G^{(1)}_{K_1}(\mathbf{x})$, and is an integral of (B.11). But then

$$G^{(1)}_{K_1}(\mathbf{x}) = \Phi^{(1)}\left(F_N(\mathbf{x})\right),$$

where $\Phi^{(1)}\colon \mathbb{R} \to \mathbb{R}$ is some smooth function.

The function

$$G^{(2)}(\mathbf{x},\varepsilon) = G^{(1)}(\mathbf{x},\varepsilon) - \Phi^{(1)}\left(F(\mathbf{x},\varepsilon)\right)$$

is an integral of (B.24) and has a second order degree of smallness in ε.

But then

$$G^{(2)}_{K_1}(\mathbf{x}) = \lim_{\varepsilon\to 0} \varepsilon^{-2} G^{(2)}(\mathbf{x},\varepsilon)$$

will be a quasihomogeneous integral of the truncated system (B.11), from which it follows that

$$G^{(2)}_{K_2}(\mathbf{x}) = \Phi^{(2)}\left(F_N(\mathbf{x})\right),$$

for some smooth function $\Phi^{(2)}\colon \mathbb{R} \to \mathbb{R}$.

Repeating this process ad infinitum we obtain that

$$G(x,\varepsilon) = \sum_{m=0}^{\infty} \Phi^{(m)}\left(F(\mathbf{x})\right).$$

But this indicates that there exists a function $H\colon \mathbb{R} \to \mathbb{R}$, such that (B.22) holds.

The lemma is proved.

If system (B.8) has an additional integral $G(\mathbf{x})$ then, according to Remark B.2, the linear system (B.20) has two homogeneous integrals

$$\phi_{(N+1)}(u^0,\mathbf{u}) = (u^0)^N \langle \mathbf{b},\mathbf{u}\rangle, \quad \psi_{(s+K)}(u^0,\mathbf{u}) = (u^0)^K \varphi_{(s)}(u),$$
$$\varphi_{(s)}(\mathbf{u}) = D^{(s)} G_K(x_0,\mathbf{u}).$$

As has already been noted, we may assume that the first basis vector $\mathbf{e_1} = (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0})$ is an eigenvector of the Kovalevsky matrix $\mathbf{K}$ with eigenvalue $\rho_1 = -1$. With this assumption we have:

Lemma B.6. *The integral $\psi_{(s+K)}(u^0, \mathbf{u})$ of the linear system (B.20) does not depend on the variable u^1.*

Proof. Consider the quasihomogeneous integral $G_K(\mathbf{x})$ of the truncated system (B.11), and let $\dot{G}_K(\mathbf{x})$ denote its total time derivative using system (B.11):

$$\dot{G}_K(x) = \langle dG_K(\mathbf{x}), \mathbf{f}_{\mathrm{q}}(\mathbf{x})\rangle .$$

By the assumptions made above respecting the smoothness of the vector field $\mathbf{f}(\mathbf{x})$ and the properties of the matrix $\mathbf{G}$ determining the quasihomogeneous structure, $\dot{G}_K(\mathbf{x})$ is a polynomial in $\mathbf{x}$ that is *identically* equal to zero. Therefore,—by the Leibniz rule—for any natural number j,

$$D^{(j)}\dot{G}_K(\mathbf{x_0}, \mathbf{u}) = \sum_{i=0}^{j} C_j^i \left\langle D^{(j-1)}(d\dot{G}_K)(x_0, u), D^{(j)}\mathbf{f}_{\mathrm{q}}(\mathbf{x_0}, \mathbf{u})\right\rangle \equiv 0,$$

where $C_j^i = \dfrac{j!}{i!(j-i)!}$.

Since the differential operators D and d clearly commute, by taking $j = s$, where s is the smallest integer for which $D^{(s)}G_K(\mathbf{x_0}, \mathbf{u}) \not\equiv 0$, we obtain

$$D^{(s)}\dot{G}_K(\mathbf{x_0}, \mathbf{u}) = \left\langle d\left(D^{(s)}\dot{G}_K(\mathbf{x_0}, \mathbf{u})\right), \mathbf{f}_{\mathrm{q}}(\mathbf{x_0})\right\rangle = \left\langle d\varphi_{(\mathbf{s})}(u), \mathbf{p}\right\rangle \equiv 0.$$

From this, taking into account that $\mathbf{p} = p\mathbf{e_1} = (p, 0, \ldots, 0)$, we obtain

$$\frac{\partial \varphi_{(s)}}{\partial u^1}(\mathbf{u}) \equiv 0,$$

and the lemma is proved.

As discussed above, we may suppose that $\mathbf{b} = (0, \ldots, 0, b)$ is an eigenvector of the Kovalevsky matrix $\mathbf{K}$ with eigenvalue $\rho_n = Q$. We have

Lemma B.7. *Let the quasihomogeneous integral $F_N(\mathbf{x})$ of the truncated system (B.11) be nonsingular for the solution of this system (B.14), i.e. $dF_N(\mathbf{x_0}) \neq \mathbf{0}$. If this system has a quasihomogeneous integral $G_K(\mathbf{x})$ that is functionally independent of $F_N(\mathbf{x})$, then the linear system (B.20) has a nontrivial homogeneous integral $\tilde{\psi}_{l+R}(u^0, \mathbf{u})$, independent of u^1 and u^n, of the following form:*

$$\tilde{\psi}_{l+R}(u^0, \mathbf{u}) = (u^0)^R \tilde{\varphi}_{(l)}(u^2, \ldots, u^{n-1}), \tag{B.27}$$

where R is some (possibly negative) integer, and $\tilde{\varphi}_{(l)}$ is a homogeneous form in the variables $u^2, \ldots, u^n$ of positive integral degree l.

Proof. We rewrite the integral $\psi_{l+R}(u^0, \mathbf{u})$ in the following way:

$$\psi_{s+K}(u^0, \mathbf{u}) = (u^{\mathbf{0}})^{\mathbf{K}} \sum_{\mathbf{j=0}}^{\mathbf{s}} \tilde{\varphi}_{(s-j)}(u^2, \ldots, u^{n-1})(u^n)^j, \tag{B.28}$$

where $\tilde{\varphi}_{(s-j)}$ is a homogeneous form of degree $s - j$ in its arguments.

The integral of system (B.20) has the form:

$$\phi_{(N+1)}(u^0, \mathbf{u}) = (u^0)^N \langle \mathbf{b}, \mathbf{u} \rangle = b(u^0)^N u^n \equiv \mathbf{c} = \text{const}.$$

Therefore we can rewrite (B.28) in the form

$$\psi(u^0, \mathbf{u}, c) = \sum_{j=0}^{s} (u^0)^{K-jN} \tilde{\varphi}_{(s-j)}(u^2, \ldots, u^{n-1}) \left(\frac{c}{b}\right)^j.$$

Since the constant c is arbitrary, each coefficient

$$\tilde{\psi}_{(s+K-(N+1)j)}(u^0, \mathbf{u}) = (\mathbf{u^0})^{\mathbf{K-jN}} \tilde{\varphi}_{(s-j)}(u^2, \ldots, u^{n-1})$$

of $\left(\frac{c}{b}\right)^j$ is an integral of system (B.20) having the required form.

However, a theoretically possible situation is where all the homogeneous polynomials $\tilde{\varphi}_{(s-j)}$, $j = 0, \ldots, s-1$ are identically equal to zero, i.e. the integral (B.28) of system (B.20) has the form

$$\psi_{(s+K)}(u^0, \mathbf{u}) = a_s (u^0)^K (u^n)^s.$$

Since $Q = \alpha N$ is the last eigenvalue of the Kovalevsky matrix, it follows immediately that

$$K = sN.$$

Consider another integral of the truncated system (B.11):

$$G_K^*(\mathbf{x}) = G_K(\mathbf{x}) = \frac{a_s}{s!} \left(b^{-1} F_N(\mathbf{x})\right)^s,$$

which likewise is a quasihomogeneous function of degree K.

The Taylor series expansion of $G_K^*(\mathbf{x})$ in the neighborhood of $\mathbf{x} = \mathbf{x_0}$ begins with terms of order at least $s + 1$, so that the linear system (B.20) has a homogeneous integral of the form

$$\psi^*_{(s_1+K)}(u^0, \mathbf{u}) = (u^0)^K \varphi^*_{(s_1)}(\mathbf{u}),$$
$$\varphi^*_{(s_1)}(\mathbf{u}) = D^{(s_1)} G_K^*(\mathbf{x}_0)\mathbf{u}, \quad s < s_1 < +\infty.$$

With this integral we can perform the procedure described above. If this integral likewise has the form

$$\psi^*_{(s_1+K)}(u^0, \mathbf{u}) = a_{s_1}(u^0)^K (u^n)^{s_1},$$

then

$$K = s_1 N,$$

which contradicts the condition $s_1 > s$.

But if $s_1 = +\infty$, then $G^*_K(\mathbf{x}) \equiv 0$. This means that $G_K(\mathbf{x})$ is the function $F_N(\mathbf{x})$ and we have obtained a contradiction.

The lemma is proved.

We continue the proof of Theorem B.2. If the truncated system (B.11) has a quasihomogeneous integral $G_K(\mathbf{x})$ that is functionally independent of $F_N(\mathbf{x})$, then according to Lemmas B.6 and B.7 the linear system (B.20) possesses a nontrivial integral of the form (B.27). Then the functions $\tilde{\varphi}_{(l)}(u^2, \dots, u^{n-1})$ must satisfy the following first order partial differential equation

$$\begin{aligned} &-\alpha R\tilde{\varphi}_{(l)}(\mathbf{u}) + \langle d\tilde{\varphi}_{(l)}(\mathbf{u}), \mathbf{K}\mathbf{u}\rangle = \\ &= -\alpha R\tilde{\varphi}_{(l)}(\mathbf{u}) + \sum\nolimits_{j=2}^{n-1} \rho_j u^j \frac{\partial \tilde{\varphi}_{(l)}}{\partial u^j}(u) = 0. \end{aligned} \tag{B.29}$$

Writing, as before, the homogeneous function $\tilde{\varphi}_{(l)}(u^2, \dots, u^{n-1})$ as a sum of elementary monomials

$$\tilde{\varphi}_{(l)}(u^2, \dots, u^{n-1}) = \sum_{k_2+\ldots+k_{n-1}=l} \tilde{\varphi}_{k_2\ldots k_{n-1}}(u^2)^{k_2}\ldots(u^{n-1})^{k_{n-1}},$$

from Eq. (B.29) we obtain that any nonzero coefficient $\tilde{\varphi}_{k_2\ldots k_{n-1}}$ in the above expansion must satisfy the resonance relation

$$-R + (q-1)\sum_{j=2}^{n-1} k_j \rho_j = 0,$$

which can be interpreted as a condition of the form (B.21), since $\rho_1 = -1$.

We need only apply Lemma B.5 to complete the proof.

The theorem is proved.

Unfortunately, the result obtained in not applicable to Hamiltonian systems, which are the most frequently encountered systems in mechanics. As shown in the paper [197], Hamiltonian systems necessarily have resonances of a special type between *pairs* of eigenvalues of the Kovalevsky matrix. On the other hand, it follows from results from the paper [109] that these resonances are not so much consequences of the presence of integrals—Hamiltonian functions—as they are consequences of the presence of invariant measures.

We apply our results to a concrete system of equations.

Example B.1. We consider the two-dimensional logistical system [189]

$$\dot{x} = x(a + cx + dy), \quad \dot{y} = y(b + ex + fy) \tag{B.30}$$

and find arithmetical conditions that must be satisfied by the coefficients of (B.30) so that the system considered not have smooth integrals represented in the form of nontrivial formal Maclaurin series in the neighborhood of the origin $x = y = 0$.

Of course, this autonomous two-dimensional system with analytic right sides cannot display chaotic behavior and must be integrable in the defined sense. However, under certain conditions, the integrals of this system will in some sense be "bad".

First of all, as follows from Lemma B.1, system (B.30) doesn't have integrals that can be written as formal power series in x, y if, for any $k_1, k_2 \in \mathbb{N} \cup \{0\}$, $k_1 + k_2 \geq 1$, the equation

$$k_1 a + k_2 b \neq 0,$$

holds. This may be rewritten as

$$-\frac{a}{b} \notin \mathbb{Q}^+.$$

On the other hand, system (B.30) is negative semi-quasihomogeneous. Its truncation

$$\dot{x} = x(cx + dy), \quad \dot{y} = y(ex + fy) \tag{B.31}$$

is a homogeneous system of order 2.

The truncated system (B.31) has an affine solution in the form of a ray

$$x_{(0)}(t) = \frac{x_0}{t}, \quad x_{(0)}(t) = \frac{y_0}{t}, \quad x_0 = \frac{d - f}{cf - ed}, \quad y_0 = \frac{e - c}{cf - ed},$$

provided that $cf - ed \neq 0$.

It is not difficult to compute the eigenvalues of the Kovalevsky matrix:

$$\rho_1 = -1, \quad \rho_2 = \rho = \frac{(d - f)(e - c)}{cf - ed}.$$

Consequently, according to Theorem B.1, system (B.30) has no polynomial integrals if

$$k_1 \neq k_2 \rho$$

for any $k_1, k_2 \in \mathbb{N} \cup \{0\}$, $k_1 + k_2 \geq 1$, i.e. if

$$\frac{(d - f)(e - c)}{cf - ed} \notin \mathbb{Q}^+.$$

Example B.2. We consider a system of equations describing the perturbed Oregonator [186]:

$$\begin{aligned} \dot{x} &= a(y - xy + x - \varepsilon xz - bx^2), \\ \dot{y} &= a^{-1}(-y - xy + cz), \\ \dot{z} &= d(x - \varepsilon xz - z). \end{aligned} \tag{B.32}$$

This system of equations describes a certain hypothetical chemical reaction of the Belousov-Zhabotinsky type, where the variables x, y, z denote concentrations of reagents and the constants a, b, c, d, ε are reaction coefficients, where $\varepsilon > 0$ is sufficiently small. Chronologically, the unperturbed case ($\varepsilon = 0$) was the first considered [50]. It turns out, even for very small values of ε, that the behavior of the system differs sharply from that of the unperturbed system. It was discovered that chaotic behavior is possible in system (B.32).

We will find arithmetic relations which, when satisfied, would guarantee that system (B.32) does not have polynomial integrals or even integrals that can be represented as formal Maclaurin series.

Explicit formulae for the eigenvalues of the matrix for the linear approximating system are quite unwieldy, so that Lemma B.1 isn't of much use as a means for establishing nonintegrability.

We apply Theorem B.1. System (B.32) is negative semi-quasihomogeneous. Its truncation

$$\dot{x} = -ax(y + bx), \quad \dot{y} = -a^{-1}xy, \quad \dot{z} = dx(1 - \varepsilon z) \tag{B.33}$$

is a quasihomogeneous system of order $q = 2$ with indices $s_x = s_y = 1, s_z = 0$.

System (B.33) has a real particular solution of quasihomogeneous ray type:

$$\begin{gathered} x_{(0)}(t) = \tfrac{x_0}{t}, \quad y_{(0)}(t) = \tfrac{y_0}{t}, \quad z_{(0)}(t) = z_0, \\ x_0 = a, \quad y_0 = a^{-1} - ab, \quad z_0 = \varepsilon^{-1}. \end{gathered}$$

The eigenvalues of the Kovalevsky matrix are:

$$\rho_1 = -1, \quad \rho_2 = 1 - a^2 b, \quad \rho_3 = -ad\varepsilon.$$

By Theorem B.1, the system of equations (B.32) doesn't have polynomial integrals when

$$-k_1 + (1 - a^2 b)k_2 - ad\varepsilon k_3 \neq 0,$$

where $k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}$, $k_1 + k_2 + k_3 \geq 1$.

Example B.3. By way of an example that illustrates Theorem B.2, we consider the problem of the integrability of the Euler-Poincaré equations over a Lie algebra [152], which has been studied in various fields of mathematical physics. In the occidental literature these equations likewise are often called the Poincaré -Arnold equations in connection with the paper [4] of V.I. Arnold. They are natural generalizations of the famous Euler equations from dynamics, which describe the inertial motion of a solid body from a point at rest and can be written in the following way:

$$\dot{m}_l = \sum_{i,j=1}^{n} C_{il}^{j} \omega^i m_j, \quad l = 1, \ldots, n. \tag{B.34}$$

The constants $\{C_{il}^j\}_{i,j,l=1}^n$ are so-called structure constants of some n–dimensional Lie algebra $\mathfrak{g}$, and $\boldsymbol{\omega} = (\omega^1, \ldots, \omega^n)$ is the "angular velocity" vector that is connected with the momentum covector $\mathbf{m} = (m_1, \ldots, m_n)$ by

$$m_j = \sum_{i=1}^{n} I_{ij} \omega^i,$$

where $\{I_{ij}\}_{i,j=1}^n$ is a positive definite symmetric tensor of type $(0, 2)$, analogous to the energy tensor for an ordinary solid body.

In the usual situation of the motion of a solid body, we have $\mathfrak{g} = \mathfrak{so}(3)$.

Equation (B.34) always have an "energy integral":

$$T = \frac{1}{2} \sum_{i,j=1}^{n} I_{ij} \omega^i \omega^j. \tag{B.35}$$

As usual, the question arises as to whether Eq. (B.34) has additional integrals that are functionally independent of (B.35). A good bit of research has been devoted to this and related problems (see e.g. the monographs [51, 112, 185] along with the literature cited there). The publications [107, 112] study the closely related question of the existence of an integral invariant (an invariant measure with an infinitely smooth density function) for the equations (B.34). Exhaustive answers to the question posed have been given in the cited papers in the case of low dimensions. It turns out that, for $n = 3$, only the *solvable* Lie algebras provide us examples of Euler-Poincaré systems without integral invariants. It is entirely possible that the solvability of the algebra $\mathfrak{g}$ will turn out to be a general obstacle to integrability.

Thus we limit ourselves to the case of solvable algebras with $n = 3$.

According to [89] there exists a canonical basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ such that the only nonzero structure constants C_{il}^j are

$$C_{13}^1 = -C_{31}^1 = \alpha, \quad C_{13}^2 = C_{31}^2 = \beta,$$
$$C_{23}^1 = -C_{32}^1 = \gamma, \quad C_{23}^2 = C_{32}^2 = \delta.$$

We likewise impose a nondegeneracy condition of the structure constants of the algebra $\mathfrak{g}$:

$$\alpha\delta - \beta\gamma \neq 0.$$

So as to keep the "angular velocity" in analogy with the dynamics of a solid body, we use the notation $\boldsymbol{\omega} = (p, q, r)$. For the "inertial tensor" $\mathbf{I}$ and its inverse $\mathbf{I}^{-1}$ we introduce the following notation:

$$\mathbf{I} = \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}, \quad \mathbf{I}^{-1} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

We let $\mathbf{m} = (x, y, z)$ denote the vector of "moments of momentum". Then the corresponding Euler-Poincaré equations assume the form:

$$\begin{aligned} \dot{x} &= -r(\alpha x + \beta y), \\ \dot{y} &= -r(\gamma x + \delta y), \\ \dot{z} &= p(\alpha x + \beta y) + q(\gamma x + \delta y), \end{aligned} \tag{B.36}$$

where $p = ax + dy + ez$, $q = dx + by + fz$, $r = ex + fy + cz$.

The integral of energy can be written in the following way:

$$\begin{aligned} T &= \tfrac{1}{2}(xp + yq + zr) \\ &= \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2 + 2Dpq + 2Epr + 2Fqr) \\ &= \tfrac{1}{2}(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz). \end{aligned} \tag{B.37}$$

The system (B.36) is homogeneous with a quadratic right side, so that we may regard it as quasihomogeneous and semi-quasihomogeneous. The integral (B.37) is, of course, nondegenerate for any vector (x_0, y_0, z_0).

We can show rather easily that Eq. (B.36) has a linear particular solution in the form of a ray:

$$x_{(0)}(t) = \frac{x_0}{t}, \quad y_{(0)}(t) = \frac{y_0}{t}, \quad z_{(0)}(t) = \frac{z_0}{t},$$

where (x_0, y_0, z_0) are certain (in general complex) numbers and $|x_0|^2 + |y_0|^2 + |z_0|^2 \neq 0$. However, in order to compute the eigenvalues of the Kovalevsky matrix, we need to use the following technical device. The matrix $\mathbf{K}$ in general has the form:

$$\mathbf{K} = \begin{pmatrix} 1 - eX - \alpha w & -fX - \beta w & -cX \\ -eY - \gamma w & 1 - fY - \delta w & -cY \\ aX + dY + \alpha u + \gamma v & dX + bY + \beta u + \delta v & 1 + eX + fY \end{pmatrix},$$

where we have introduced the notations

$$u = ax_0 + dy_0 + ez_0, \quad v = dx_0 + by_0 + fz_0, \quad w = ex_0 + fy_0 + cz_0,$$
$$X = \alpha x_0 + \beta y_0, \quad Y = \gamma x_0 + \delta y_0.$$

Since Eq. (B.36) has a quadratic energy integral (B.37), we have a priori knowledge of *two* roots of the characteristic polynomial $\Delta_K(\rho)$:

$$\rho_1 = -1, \quad \rho_3 = 2.$$

Consequently the quadratic trinomial

$$P(\rho) = \rho^2 - \rho - 2$$

divides the polynomial $\Delta_K(\rho)$.

Looking at the matrix **K**, we can compute the remainder upon division of $\Delta_K(\rho)$ by $P(\rho)$:

$$Q(\rho) = \rho - 2 + (\alpha + \delta)w.$$

Consequently

$$\rho = \rho(w) = 2 - (\alpha + \delta)w,$$

so that, in determining ρ_2, it suffices to compute the quantity w.

For this we need to pay attention to the fact that the equation for determining x_0, y_0 "almost" has the closed form:

$$(1 - \alpha w)x_0 - \beta w y_0 = 0, \quad -\gamma w x_0 + (1 - \delta w)y_0 = 0.$$

This linear homogeneous system has a nontrivial solution if and only if its determinant equals zero. From this we get

$$w = \frac{1}{2}\frac{\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{\alpha\delta - \beta\gamma}. \tag{B.38}$$

If

$$\rho(w) \in \mathbb{Q},$$

where the quantity w is given by the equalities (B.38), then the eigenvalues of the Kovalevsky matrix don't satisfy resonance relations of type (B.24) and the Euler-Poincaré equations (B.36) don't have an integral that is representable as a nontrivial formal power series in x, y, z or in p, q, r, and that is functionally independent of (B.37).

We look at two interesting particular cases. Let $\alpha = \delta = 0$. Then

$$\rho_2 = 2,$$

and (B.36) becomes suspect for integrability.

In fact, in this case there is an obvious additional first integral:

$$G = \frac{1}{2}(\gamma x^2 - \beta y^2).$$

It is interesting to note that if $\beta\gamma < 0$, then, as shown in [107, 112] the system considered has an integral invariant (the Lie group that generates the Lie algebra $\mathfrak{g}$ is unimodular). If, conversely, $\beta\gamma > 0$, then there is no integral invariant but there exists an invariant measure, whose density function has the desired high (but finite) order of smoothness.

Now let $\beta = \gamma = 0$. Then

$$\rho_2 = 1 - \frac{\alpha}{\delta} \quad \text{or} \quad 1 - \frac{\delta}{\alpha},$$

and the condition for integrability assumes the form

$$\frac{\alpha}{\delta} \notin \mathbb{Q}.$$

If $a\delta > 0$, then it follows from [107, 112] that the Euler-Poincaré system (B.36) does not possess an invariant measure even with summable density.

In the paper [88] the Euler-Poincaré equation is studied on a multidimensional solvable Lie algebra. It is shown that, independent of the structure of the inertia tensor for a typical non-nilpotent solvable Lie algebra, the Euler-Poincaré equation branches in the plane of complex time.

In conclusion we will briefly discuss the conditions for nonintegrability obtained in Theorems B.1 and B.2 and compare them with Yosida's criterion [197].

According to Yoshida's criterion, if at least one of the eigenvalues of the Kovalevsky matrix is irrational, then the quasihomogeneous system of differential equations is *algebraically nonintegrable*. This assertion admits a complex-analytic interpretation in the spirit of a paper of S.L. Ziglin [202]. The basic idea of that paper is that the branching of solutions in the complex plane impedes integrability.

Consider a Fuchsian system of linear differential equations, obtained from system (B.16) by linearization in the neighborhood of $\mathbf{u} = \mathbf{0}$:

$$t\dot{\mathbf{u}} = \mathbf{K}\mathbf{u}. \tag{B.39}$$

The monodromy operator, acting on the space of solutions of this system, has the matrix

$$\mathbf{M} = \exp(2\pi i \mathbf{K})$$

which, if the hypothesis of Yoshida's criterion is satisfied, cannot be a rational root of the identity matrix $\mathbf{E}$.

This means that system (B.39) has a particular solution with an unbounded Riemann surface. But if either of the inequalities (B.15) or (B.21) is satisfied, then the space of solutions of system (B.39) has a still more complicated structure.

As follows from Lemma B.4, any $n - 1$ quasihomogeneous integrals of the truncated system (B.11) are functionally dependent at the point $\mathbf{x} = \mathbf{x_0}$ if the hypothesis of Yoshida's criterion is satisfied. It is, however, unclear—as is noted in the paper [69]—whether this domain of dependency can be extended. The hypothesis of Theorems B.1 and B.2 impose substantially harsher requirements on the properties of the eigenvalues of the Kovalevsky matrix, which allow us to formulate a stronger assertion about nonintegrability.

Literature

Translator's Note: English translations are cited when available and *Mathematical Reviews* and/or *Zentralblatt für Mathematik* references are given for untranslated books and articles, when available, e.g. MR 01234567 and/or Zbl 0123.45678. In transcription of most Russian words and proper names, the BGN/PCGN system is followed. Exceptions occur when a name has an accepted spelling in English (e.g. *Kovalevsky*) or when another spelling is used in a publication (e.g. *Arnold* as in [6]). The name *Bohl* has the Russian form *Bol'* in [23], as well as the Latin form in [22].

The language of publication is English except as otherwise noted or as is obvious from the title (for French, German and Italian titles). For the original version of translated *books*, only the place and year of publication and the name of the publisher is noted, e.g. "Publication source: Moscow: Editorial URSS (2002)". For translated *articles*, the exact journal reference of the original is given, but not the original title.

For books and journals published by Springer-Verlag, the place of publication will be understood to be New York-Berlin-Heidelberg and will be omitted.

1. Ablowitz, M.J., Ramani, A., Segur, H.: A connection between nonlinear evolution equations and o.d.e.'s of P-type. J. Math. Phys. **21**, 715–721 (1980)
2. Appel'rot, G.G.: On §2 of S.V. Kovalevsky's memoir Sur le problème de la rotation d'un corps solide autour d'un point fixe. Mat. Sb. **16**(1), 483–507, 592–596 (1892)
3. Arnol'd, V.I.: Small denominators and problems of stability of motion in classical and celestial mechanics. Russ. Math. Surv. **18**(6), 85–191 (1963). Translated from: Usp. Mat. Nauk. **18**(6) (114), 91–192 (1963)
4. Arnold, V.I.: Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Ann. Inst. Fourier **16**(1), 319–366 (1966)
5. Arnol'd, V.I.: Algebraic unsolvability of the problem of Lyapunov stability and the problem of the topological classification of the singular points of an analytic system of differential equation. Funct. Anal. Appl. **4**, 173–180 (1970). Translated from: Funkts. Anal. Prilozh. **4**(3), 1–9 (1970)
6. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Springer, New York (1978). Publication source: Nauka, Moscow (1974)
7. Arnold, V.I.: Geometrical Methods in the Theory of Ordinary Differential Equations. Springer, New York (1988). Publication source: Nauka, Moscow (1978)

V.V. Kozlov and S.D. Furta, *Asymptotic Solutions of Strongly Nonlinear Systems of Differential Equations*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-642-33817-5,

8. Arnol'd, V.I., Il'yashenko, Y.S.: Ordinary differential equations. Dynamical systems I. In: Encyclopedia of Mathematical Sciences, vol. 1, pp. 1–148. Springer, Berlin (1988). Translated from: Itogi Nauki i Tekhniki, pp. 7–149, 244 in series Contemporary Problems in Mathematics. Fundamental Directions, vol. 1 (1985)
9. Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Mathematical aspects of classical and celestial mechanics. In: Encyclopedia of Mathematical Sciences, vol. 3. Springer, Berlin (1993). Tranlated from: Itogi Nauki i Tekhniki in series Contemporary Problems in Mathematics. Fundamental Directions, vol. 1 (1985)
10. Bardin, B.S.: On the motion of satellites, asymptotic to their regular precession (Russian). Kosmich. Isled. **29**(6), 822–827 (1991)
11. Bardin, B.S.: Asymptotic solutions of Hamiltonian systems in the case of first-order resonance. J. Appl. Mech. **55**(4), 469–474 (1991). Translated from: Prikl. Mat. Mekh. **55**(4), 587–593 (1991)
12. Belaga, E.G.: On the reducibility of a system of ordinary differential equations in the neighborhood of a conditionally periodic motion. Sov. Math. Dokl. **3**(2), 360–364 (1962). Translated from: Dokl. Akad. Nauk. SSSR **143**, 255–258 (1962)
13. Belitskiy, G.R.: Normal Forms, Invariants and Local Mappings (Russian). Naukova Dumka, Kiev (1979). MR 537763, Zbl 0479.58001
14. Bellman, R.: Stability Theory of Differential Equations. McGraw-Hill, New York (1953)
15. Bender, C.M., Orszag, S.A.: Advanced Mathematical Methods for Scientists and Engineers. Springer, New York (1999)
16. Birkhoff, G.D.: The restricted problem of three bodies. Palermo Rend. **39**, 1–115 (1915)
17. Birkhoff, G.D.: Dynamical Systems. American Mathematical Society, Providence (1966)
18. Bogolyubov, N.N., Mitropolsky, Y.O.: Asymptotic Methods in the Theory of Non-linear Oscillations. Gordon and Breach, New York (1961)
19. Bogolyubov, N.N., Mitropolskiy, Y.O., Smoylenko, A.M.: Methods of Accelerated Convergence in Nonlinear Mechanics. Springer, New York (1976). Publication source: Naukova Dumka, Kiev (1969)
20. Bogoyavlenskiy, A.A.: On some special solutions of the problem of motion of a heavy rigid body about a fixed point. J. Appl. Math. Mech. **22**, 1049–1064 (1958)
21. Bogoyavlenskiy, A.A.: On particular cases of motion of a heavy rigid body about a fixed point. J. Appl. Math. Mech. **22**, 873–906 (1958)
22. Bohl, P.: Über die Bewegung eines mechanischen systems in der Nähe einer Gleichgewichtslage. J. Reine Angew. Math. **127**, 179–276 (1904)
23. Bol', P.G.: Selected Works (Russian), Riga: Akad. Nauk. Latv. SSR (1961). MR 0354285
24. Bolotin, S.V., Negrini. P.: Asymptotic trajectories of gyroscopic systems (Russian). Vestn. Mosk. Univ. Ser. I, (6), 66–75 (1993). MR 1358122, Zbl 0842.70005
25. Bolotin, S., Negrini, P.: Asymptotic solutions of Lagrangian systems with gyroscopic forces. Nonlinear Differ. Equ. Appl. **2**(4), 417–444 (1995)
26. Bountis, T., Segur, H., Vivaldi, F.: Integrable Hamiltonian systems and Painlevé property. Phys. Rev. A **25**, 1257–1264 (1982)
27. Bountis, T.C., Drossos, L.B., Lakshmanan, M., Parthasarathy, S.: On the non-integrability of a family of Duffing – van der Pol oscillators. J. Phys. A **26**(23), 6927–6942 (1993)
28. Briot, C., Bouquet, T.: Recherches sur les proprietés des équations différentielles. J. École Polytech. **21**(36), 133–199 (1856)
29. Bryuno, A.D.: Power asymptotics of solutions of nonlinear systems (Russian). Izv. Akad. Nauk. SSSR Ser. Mat. **29**, 329–364 (1965). Zbl 0133.33604
30. Bryuno, A.D.: The analytic form of differential equations I, II (Russian). Trudy Moskov. Mat. Obshch. **25**, 119–262 (1971); **26**, 199–239 (1972). English translation of part II only: Trans. Mosc. Math. Soc. **26** (1972), 199–239 (1974). Zbl 0283.34013
31. Bryuno, A.D.: Stability in a Hamiltonian system. Math. Notes **40**, 726–730 (1986). Translated from Mat. Zametki **40**(3), 385–392 (1986)
32. Bryuno, A.D.: A Local Method of Nonlinear Analysis for Differential Equations, 2nd edn. Springer, Berlin (1989). Publication source: Nauka, Moscow (1979)

33. Bugaev, N.V.: Beginning of largest and smallest exponents in the theory of differential equations. Total and partial integrals (Russian). Mat. Sb. **16**(1), 9–80 (1982)
34. Bulatovich, R., Kazhich, M.: Asymptotic solutions of equations of motion of nonholonomic Chaplygin systems (Russian). Mathematica Montesnigri **2**, 11–20 (1993). MR 1284893
35. Bykov, Y.V.: On Certain Problems in the Theory of Integro-Differential Equations (Russian). Akademii Nauk Kirgizskoi SSR, Frunze (1957)
36. Chang, Y.F., Tabor, M., Weiss, J.: Analytic structure of the Henon-Heiles Hamiltonian in integrable and nonintegrable regimes. J. Math. Phys. **23**(4), 531–538 (1982)
37. Chang, Y.F., Greene, J.M., Tabor, M., Weiss, J.: The analytic structure of dynamical systems and self-similar natural boundaries. Phys. D **8**(1–2), 183–207 (1983)
38. Chebotarev, N.G.: Newton's Polygon and Its Role in the Contemporary Development of Mathematics (Russian). In: Isaac Newton. Akademii Nauk SSSR, Moscow/Leningrad (1943)
39. Chetaev, N.G.: The Stability of Motion. Papers on Analytical Mechanics (Russian). Akademii Nauk SSSR, Moscow (1962)
40. Chetayev, N.G.: The Stability of Motion. Pergamon, Oxford/New York/Paris (1961)
41. Chow, S.-N., Hale, J.K.: Methods of Bifurcation Theory. Springer, New York (1982)
42. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955)
43. Cushman, R.H.: A survey of normalization techniques applied to perturbed Keplerian systems. Dyn. Rep. Expos. Dyn. Syst. New Ser. **1**, 54–112 (1992)
44. Demidovich, B.P.: Lectures on the Mathematical Theory of Stability (Russian). Nauka, Moscow (1967). MR 0226126, Zbl 0155.41601
45. Deprit, A.: Canonical transformations depending on a small parameter. Celest. Mech. **1**, 12–30 (1969)
46. Duboshin, G.N.: Celestial Mechanics: Basic Problems and Methods (Russian). Nauka, Moscow (1975)
47. El'sgol'ts, L.E., Norkin, S.B.: Introduction to the Theory and Application of Differential Equations with Deviating Arguments. Academic, New York/London (1973)
48. Engel, U.M., Stegemerten, B., Ecklet, P.: Normal forms and quasi-integrals for the Hamiltonians of magnetic bottles. J. Phys. A **28**, 1425–1448 (1995)
49. Ercolani, N., Siggia, E.D.: Painlevé property and integrability. Phys. Lett. A **119**(3), 112–116 (1986)
50. Field, R.J., Noyes, R.M.: Oscillations in chemical systems IV: limit cycle behavior in a model of real chemical reaction. J. Chem. Phys. **60**(5), 1877–1884 (1974)
51. Fomenko, A.T., Trofimov, V.V.: Integrable Systems on Lie Algebras and Symmetric Spaces. Gordon and Breach, London/New York (1988)
52. Fournier, J.D., Levine, G., Tabor, M.: Singularity clustering in the Duffing oscillator. J. Phys. A **21**, 33–54 (1988)
53. Furta, S.D.: On the asymptotic solutions of the equations of motion of mechanical systems. J. Appl. Math. Mech. **50**, 726–731 (1986). Translated from: Prikl. Mat. Mekh. **50**, 938–944 (1986)
54. Furta, S.D.: Instability of the equilibrium states of nonnatural conservative systems (Russian). In: The Method of Lyapunov Functions in the Analysis of the Dynamics of Systems, pp. 203–206. Nauka, Novosibirsk (1987)
55. Furta, S.D.: The instability of equilibrium of a gyroscopic system with two degrees of freedom (Russian). Vestn. MGU, Ser. I Mat. Mekh. (5), 100–101 (1987). MR 913275
56. Furta, S.D.: Asymptotic motions of nonsymmetric solid bodies in circular orbit in the presence of third-order resonance (Russian). Kos. Issled. **26**(6), 943–944 (1988)
57. Furta, S.D.: Asymptotic trajectories of mechanical systems and the problem of the inversion of the Routh theorem on stability of equilibrium (Russian). In: Differential and Integral Equations: Methods of Topological Dynamics, pp. 80–86. GGU, Gor'kiy (1989). Zbl 0800.35002
58. Furta, S.D.: Asymptotic trajectories of natural systems under the action of forces of viscous friction (Russian). In: Analytic and Numerical Methods for Studying Mechanical Systems, pp. 35–38. MAI, Moscow (1989)

59. Furta, S.D.: Asymptotic solutions of autonomous systems of ordinary differential equations in the critical case of several zero eigenvalues (Russian). In: Problems in the Mechanics of Controlled Motion. Nonlinear Dynamical Systems, pp. 152–156. PGU, Perm' (1989)
60. Furta, S.D.: Asymptotic solutions of semi-quasihomogeneous systems of differential equations (Russian). In: Mathematical Methods in Mechanics, pp. 93–96. MGU, Moscow (1990)
61. Furta, S.D.: Asymptotic trajectories of dynamical systems (Russian). In: Dynamics, Stochastics, Bifurcation, pp. 89–101. ITU, Gor'kiy (1990). MR 1156355, Zbl 0900.34054
62. Furta, S.D.: Asymptotic solutions of differential equation systems in the case of purely imaginary eigenvalues. Sov. Appl. Mech. **27**(2), 204–208 (1991). Translated from: Differ. Uravn. **26**(8), 1348–1351 (1990)
63. Furta, S.D.: Instability of the equilibrium positions of restricted mechanical systems. Sov. Appl. Mech. **27**(2), 204–208 (1991). Translated from: Prikl. Mekh. (Kiev) **27**(2), 43–46 (1991)
64. Furta, S.D.: On asymptotic solutions of differential equations with deviating argument in certain critical cases. Russ. Acad. Sci. Math. Sb. **78**(1), 175–186 (1994). Translated from: Mat. Sb. **184**(2), 43–56 (1993)
65. Furta, S.D.: On non-integrability of general systems of differential equations. Z. Angew. Math. Phys. **47**(1), 112–131 (1996)
66. Gantmacher, F.R.: The Theory of Matrices, vols. 1, 2. Chelsea, New York (1959)
67. Gilmore, R.: Catastrophe Theory for Scientists and Engineers. Wiley, New York (1981)
68. Golubev, V.V.: Lectures on the Analytic Theory of Differential Equations (Russian). Gosizdat, Moscow/Leningrad (1950). MR 0042571, Zbl 0038.24201
69. Gonzales-Gascon, F.A.: A word of caution concerning the Yoshida criterion on algebraic integrability and Kowalevsky exponents. Celest. Mech. **44**, 309–311 (1988)
70. Grodzenskiy, S.Y.: Andrey Andreyevich Markov (Russian), pp. 1865–1922. Nauka, Moscow (1987). MR 909492
71. Gromak, V.I.: Solutions of the second Painlevé equation. Differ. Equ. **18**, 537–545 (1982). Translated from: Differ. Uravn. **18**(5), 753–763, 914–915 (1982)
72. Hadamard, J.: Sur l'iteration et les solutions asymptotiques des équations différentielles. Bull. Soc. Math. Fr. **29**, 224–228 (1901)
73. Hale, J.: Theory of Functional Differential Equations, 2nd edn. Springer, New York (1977)
74. Halmos, P.R.: Finite Dimensional Vector Spaces. Springer, New York (1958)
75. Hardy, G.H.: Divergent Series. Clarendon, Oxford (1949)
76. Hartman, P.: Ordinary Differential Equations. Birkhäuser, Boston (1982)
77. Hartman, P., Wintner, A.: Asymptotic integrations of ordinary differential equations. Am. J. Math. **77**, 692–724 (1955)
78. Henon, M., Heiles, C.: The applicability of the third integral of the motion: some numerical results. Astron. J. **69**, 73–79 (1964)
79. Hess, W.: Über die Euler'schen Bewegungsgleichungen und über eine neue particuläre Lösung des Problems der Bewegung eines starren Körpers um einen festen Punkt. Math. Annal. **37**(2), 153–181 (1890)
80. Hill, G.W.: Researches in the lunar theory. In: Hill, G.W. (ed.) Collected Mathematical Works, vol. I. Carnegie Institute, Washington, DC (1905)
81. Hirsch, M., Pugh, C., Schub, M.: Invariant Manifolds. Springer, Berlin/New York (1977)
82. Hori, G.I.: Theory of general perturbations with unspecified canonical variables. J. Jpn. Astron. Soc. **18**(4), 287–296 (1966)
83. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge/New York (1991)
84. Il'yashenko, Y.C.: Analytic unsolvability of the stability problem and the problem of topological classification of the singular points of analytic systems of differential equations. Math. Sb. USSR **28**, 140–152 (1976). Translated from: Mat. Sb. **99**(2), 162–175 (1976)
85. Il'yashenko, Y.C.: Divergence of series reducing an analytic differential equation to linear normal form at a singular point. Funct. Anal. Appl. **13**, 227–229 (1980). Translated from Funkts. Anal. Prilozh. **13**(3), 87–88 (1979)

86. Il'yashenko, Y.C., Pyartli, A.S.: Materialization of resonances and divergence of normalizing series for polynomial differential equations. J. Sov. Math. **32**, 300–313 (1986). Translated from: Trudy Sem. im. G.I. Petrovskovo (8), 111–127 (1982)
87. Ince, E.L.: Ordinary Differential Equations. Dover, New York (1944)
88. Izmajlova, O.V., Kozlov, V.V.: Analytical properties of solutions of the Euler-Poincaré equations on solvable Lie algebras (Russian). Vestn. Mosk. Univ. Ser. I. Mat. Mekh. (3), 60–65 (1996). MR 1445277
89. Jacobson, N.: Lie Algebras. Wiley-Interscience, New York (1962)
90. Jahnke, E., Emde, F., Lösch, F.: Tables of Higher Functions. McGraw-Hill, New York/Toronto/London (1960)
91. Johnson, R., Sell, G.: Smoothness of spectral subbundles and reducibility of quasiperiodic linear differential systems. J. Differ. Equ. **41**, 262–288 (1981)
92. Jorba, A., Simo, C.: On the reducibility of linear differential equations with quasiperiodic coefficients. J. Differ. Equ. **98**, 111–124 (1992)
93. Kamenkov, G.V.: Selected Works in Two Volumes (Russian). Nauka, Moscow (1971/1972). MR 0469662, MR 0492597
94. Kantorovich, L.V., Akilov, G.P.: Functional Analysis (Russian), 3rd rev. edn. Nauka, Moscow (1984). MR 788496, Zbl 0555.46001
95. Karakostas, G., Gyšri, I.: Global stability in job systems. J. Math. Anal. Appl. **131**, 85–96 (1988)
96. Karapetyan, A.V., Rumyancev, V.V.: Stability of conservative and dissipative systems (Russian). Itogi Nauki i Tekhniki, Obshchaya Mekhanika **6**, 3–128 (1983). MR 766567
97. Khazin, L.G., Shnol', E.E.: Investigation of the asymptotic stability of equilibrium under 1:3 resonance (Russian). Preprint no. 67, Inst. Prikl. Mat., Akad. Nauk. SSSR (1978). MR 514794
98. Khazin, L.G., Shnol', E.E.: The simplest cases of algebraic unsolvability in problems of asymptotic stability. Sov. Math. Dokl. **19**, 773–776 (1978). Translated from Dokl. Akad. Nauk. SSSR **240**(60, 1309–1311 (1978)
99. Khazin, L.G., Shnol', E.E.: Equilibrium stability conditions under 1:3 resonance. J. Appl. Math. Mech. **44**(2), 229–237 (1980)
100. Khazin, L.G., Shnol', E.E.: Stability of Critical Equilibrium States. Manchester University Press, Manchester (1991). Publication source: Nauchn. Tsentr Biol. Issled/Akad. Nauk. SSSR, Pushchino (1985)
101. Kneser, A.: Studien über die Bewegungsvorgänge in der Umgebung instabiler Gleichgewichtslagen I, II. J. Reine Angew. Math. **115**, 308–327 (1895); **118**, 186–223 (1897)
102. Kovalevsky, S.: Sur le problème de la rotation d'un corps solide autour d'un point fixé, I,II. Acta Math. **12**, 177–232 (1889); **14**, 81–93 (1890)
103. Kozlov, V.V.: Hypothesis on the existence of asymptotic motions in classical mechanics (Russian). Funkts. Anal. Prilozh. **16**(4), 72–73 (1982). MR 684135, Zbl 0505.70024
104. Kozlov, V.V.: Asymptotic solutions of equations of classical mechanics. J. Appl. Math. Mech. **46**(4), 454–457 (1983). Translated from: Prikl. Mat. Mekh. **46**, 573–577 (1982)
105. Kozlov, V.V.: On the stability of equilibria of nonholonomic systems. Sov. Math. Dokl. **33**, 654–656 (1986). Translated from: Dokl. Akad. Nauk. SSSR **288**(2), 289–291 (1986)
106. Kozlov, V.V.: Asymptotic motions and the inversions of the Lagrange-Dirichlet theorem. J. Appl. Math. Mech. **50**(6), 719–725 (1987). Translated from: Prikl. Mat. Mekh. **50**(6), 928–937 (1986)
107. Kozlov, V.V.: Invariant measures of Euler-Poincaré equations on Lie algebras. Funct. Anal. Appl. **22**(1), 58–59. Translated from: Funkts. Anal. Prilozh. **22**(1), 69–70 (1988)
108. Kozlov, V.V.: On a Kelvin problem. J. Appl. Math. Mech. **53**(1), 133–135 (1989). Translated from: Prikl. Mat. Mekh. **53**(1), 165–167 (1989)
109. Kozlov, V.V.: Tensor invariants of quasihomogeneous systems of differential equations, and the Kovalevskaya-Lyapunov asymptotic method. Math. Notes **51**(2), 138–142 (1992). Translated from: Mat. Zametki **51**(2), 46–52 (1992)
110. Kozlov, V.V.: On equilibrated nonholonomic systems (Russian). Vestn. Mosk. Univ. Ser. I Mat. Mekh. (3), 74–79 (1994). MR 1315728

111. Kozlov, V.V.: The asymptotic motions of systems with dissipation. J. Appl. Math. Mech. **58**(5), 787–792 (1994). Translated from: Prikl. Mat. Mekh. **58**(5), 31–36 (1994)
112. Kozlov, V.V.: Symmetries, Topology and Resonances in Hamiltonian Mechanics. Springer, New York (1996)
113. Kozlov, V.V.: Uniform distribution on a torus. Mosc. Univ. Math. Bull. **59**(2), 23–31 (2004). Translated from: Vestn. Mosk. Univ. Ser. I Mat. Mekh. (2), 22–29, 78 (2004)
114. Kozlov, V.V.: Gyroscopic stabilization of degenerate equilibria and the topology of real algebraic varieties. Dokl. Math. **77**(3), 412–415 (2008). Translated from Dokl. Akad. Nauk., Ross. Akad. Nauk. **420**(4), 447–450 (2008)
115. Kozlov, V.V., Furta, S.D.: On solutions with generalized power asymptotics to systems of differential equations. Math. Notes **58**(6), 1286–1293 (1995). Translated from: Mat. Zametki **58**(6), 851–861, 959 (1995)
116. Kozlov, V.V., Furta, S.D.: Lyapunov's first method for strongly non-linear systems. J. Appl. Math. Mech. **60**(1), 7–18 (1996). Translated from: Prikl. Mat. Mekh **60**(1), 10–22 (1996)
117. Kozlov, V.V., Palamodov, V.P.: On asymptotic solutions of equations of classical mechanics. Sov. Math. Dokl. **25**, 335–339 (1982). Translated from: Dokl. Akad. Nauk. SSSR **263**(2), 285–289 (1982)
118. Kozlov, V.V., Treshchev, D.V.: Kovalevskaya numbers of generalized Toda chains. Math. Notes **46**(5), 840–848 (1989). Translated from: Mat. Zametki **46**(5), 17–28 (1989)
119. Kozlov, V.V., Treshchev, D.V.: Instability of isolated equilibria of dynamical systems with invariant measure in spaces of odd dimension. Math. Notes **65**(5), 565–570 (1999). Translated from: Mat. Zametki **65**(5), 674–680 (1999)
120. Krasil'nikov, P.S.: Necessary and sufficient conditions for stability under 1:3 resonance (Russian). In: Research on Stability and Vibration Problems and their Application to the Dynamics of Aircraft, pp. 31–37. MAI, Moscow (1980)
121. Krasil'nikov, P.S.: Algebraic criteria for asymptotic stability under 1:1 resonance. J. Appl. Math. Mech. **57**(4) 583–590 (1993). Translated from: Prikl. Mat. Mekh. **57**(4), 5–11 (1993)
122. Krasil'nikov, P.S.: On extensions of the solution space of Hamilton-Jacobi equations and their application to stability problems (Russian). Preprint No. 550, Inst. Prob. Mech. Akad. Nauk. (1995)
123. Krasovskiy, N.N.: On stability in the first approximation (Russian). Prikl. Mat. Mekh. **19**, 516–530 (1955). MR 0073021
124. Krasovskiy, N.N.: Certain Problems in the Theory of Stability of Motion (Russian). Gos, Mosc. Izdat. Fiz.-Mat. Lit. (1959) MR 0106313
125. Kuznetsov, A.N.: Differentiable solutions to degenerate systems of ordinary equations. Funct. Anal. Appl. **6**, 119–127 (1972). Translated from: Funkts. Anal. Prilozh. **6**(2), 41–51 (1972)
126. Kuznetsov, A.N.: Existence of solutions entering at a singular point of an autonomous system having a formal solution. Funct. Anal. Appl. **23**(4), 308–317 (1989). Translated from: Funkts. Anal. Prilozh. **23**(4), 63–74 (1989)
127. Laloy, M.: On equilibrium instability for conservative and partially dissipative systems. Int. J. Non-Linear Mech. **11**, 295–301 (1976)
128. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields. Butterworth Heinemann, Amsterdam/Boston (2004)
129. Lavrent'ev, M.A.: Methods of the Theory of Functions of a Complex Variable (Russian). Nauka, Moscow (1987). MR 1087298
130. Lunev, V.V.: Meromorphic solutions of the equations of motion of a heavy solid with a fixed point. J. Appl. Math. Mech. **58**(1), 31–41 (1994). Translated from: Prikl. Mat. Mekh. **58**(1), 30–39 (1994)
131. Lyapunov, A.M.: Sur l'instabilité de équilibre dans certains cas où la fonction de forces n'est pas un maximum. J. Math. V **3**, 81–94 (1897)
132. Lyapunov, A.M.: On a property of the differential equations for the problem of the motion of a solid body with fixed point (Russian). In: Collected Works, vol. 1, 407–417. Akademii Nauk SSSR, Moscow/Leningrad (1954)

133. Lyapunov, A.M.: The General Problem of the Stability of Motion. Taylor & Francis, London (1992)
134. Markeev, A.P.: Stability of a canonical system with two degrees of freedom in the presence of resonance. J. Appl. Math. Mech. **32**(4), 766–772 (1968). Translated from: Prikl. Mat. Mekh. **32**(4), 738–744 (1968)
135. Markeev, A.P.: On motions asymptotic to triangular points of libration of the restricted circular three-body problem. J. Appl. Math. Mech. **51**(3), 277–282 (1987). Translated from: Prikl. Mat. Mekh. **51**(3), 355–362 (1987)
136. Markeev, A.P.: Resonances and asymptotic trajectories in Hamiltonian systems. J. Appl. Math. Mech. **54**(2), 169–173 (1990). Translated from Prikl. Mat. Mekh. **54**(2), 207–212 (1990)
137. Marsden, J.E., McCracken, M.: The Hopf Bifurcation and Its Applications. Springer, New York (1976)
138. Matveev, M.V.: Lyapunov stability of equilibrium states of reversible systems. Math. Notes **57**(1), 63–72 (1995). Translated from: Mat. Zametki **57**(1), 90–104 (1995)
139. Milnor, J.W.: Topology from the Differentiable Viewpoint. University Press of Virginia, Charlottesville (1965)
140. Molchanov, A.M.: Subdivision of motions and asymptotic methods in the theory of nonlinear oscillations (Russian). Dokl. Akad. Nauk. SSSR **2**(5), 1030–1033 (1961). MR 0119495
141. Moser, J.: Lectures on Hamiltonian Systems. American Mathematical Society, Providence (1968)
142. Narasimkhan, P.: Analysis on Real and Complex Manifolds. North-Holland, Amsterdam (1985). Publication source: Mir, Moscow (1973)
143. Nekrasov, P.A.: Analytic investigation of an instance of motion of a heavy solid body near its fixed point (Russian). Mat. Sb. **18**(1), 161–274 (1896)
144. Neymark, Y.I., Fufaev, N.A.: Dynamics of Nonholonomic Systems. Translations of Mathematical Monographs, vol. 33. American Mathematical Society, Providence (1972). Publication source: Moscow, Nauka (1967)
145. Ovsiannikov, L.V.: Group Analysis of Differential Equations. Academic, New York (1982). Publication source: Nauka, Moscow (1978)
146. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
147. Osipov, Y.S.: On the reduction principle in critical cases of stability of time-lag. J. Appl. Math. Mech. **29**, 963–974 (1965). Translated from Prikl. Mat. Mekh. **29**(5), 810–820 (1965)
148. Palamodov, V.P.: Stability of motion and algebraic geometry. In: Kozlov V.V. (ed.) Dynamical Systems in Classical Mechanics. Translations Series 2, vol. 168, pp. 5–20. American Mathematical Society, Providence (1995)
149. Peiffer, K., Carlier, P.: A remark on the inversion of Lagrange-Dirichlet's theorem. Coll. Math. Soc. Janos Bolyai 53, Qual. Theor. Differ. Equ. Szeged 473–484 (1988)
150. Perron, O.: Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzgleichungen. J. Reine Angew. Math. **161**, 41–64 (1929)
151. Poincaré, H.: Sur les propriétés des fonctions définies par les équations aux différences partielles. Gauthier-Villars, Paris (1879)
152. Poincaré, H.: Sur une forme nouvelle des équations de la mécanique. C.R. Acad. Sci. Paris **132**, 369–371 (1901)
153. Rössler, O.E.: An equation for continuous chaos. Phys. Lett. A **57**, 397–398 (1976)
154. Rouche, N., Habets, P., Laloy, M.: Stability by Ljapunov's Direct Method. Springer, New York (1977)
155. Rumyantsev, V.V.: On stability of motion of nonholonomic systems. J. Appl. Math. Mech. **31**, 282–293 (1967). Translated from: Prikl. Mat. Mekh. **31**, 260–271 (1967)
156. Rumyantsev, V.V., Sosnitskiy, S.P.: The stability of the equilibrium of holonomic conservative systems. J. Appl. Math. Mech. **57**(6), 1101–1122 (1993). Translated from: Prikl. Mat. Mekh. **57**(6), 143–166 (1993)
157. Salvadori, L.: Criteri di instabilità per i moti merostatici di un sistema olonomo. Rendiconti. Accademia delle scienze fisiche e matematiche. Società nazionale di Scienze. Lettere ed Arti di Napoli **27**(4), 535–542 (1960)

158. Salvadori, L.: Sull'estensione ai sistemi dissipativi del criterio di stabilità del Routh. Ricerche di Matematica **15**, 162–167 (1966)
159. Salvadori, L.: On the conditional total stability of equilibrium for mechanical systems. Le Mat. **46**, 415–427 (1991)
160. Salvadori, L.: Stability problems for holonomic mechanical systems. In: La mechanique analitique de Lagrange et son héritage II. Atti della Accademia delle Scienze di Torino, suppl. no. 2 al vol. 126, pp. 151–168. Classe di Scienze Fisiche, Matematiche e Naturali (1992)
161. Salvadori L., Visentin F.: Sulla stabilità totale condizionata nella meccanica dei sistemi olonomi. Rendiconti di Matematica e delle sue Applicazioni **12**, 475–495 (1992)
162. Sergeev, V.S.: Estimation of the domain of attraction for a class of integro-differential equations (Russian). In: Stability of Motion, pp. 88–93. Nauka, Novosibirsk (1985). MR 817736
163. Sergeev, V.S.: Stability of the solutions of integro-differential equations in specific cases (Russian). In: Method of Lyapunov Functions in the Analysis of Dynamical Systems, pp. 98–105. Nauka, Novosibirsk (1987). MR 956531
164. Shestakov, A.A.: Asymptotic behavior of solutions of mulidimensional systems of ordinary differential equations with a singular point of higher order. Sov. Math. Dokl. **1**, 394–397 (1960). Translated from: Dokl. Akad. Nauk. SSSR **131**(5), 1038–1041 (1960)
165. Shimanov, S.N.: Stability in the critical case of one zero root for systems with retardation (Russian). Prikl. Mat. Mekh. **24**(3), 447–457 (1960)
166. Shimanov, S.N.: Stability in the critical case of one zero root for systems with retardation (singular case) (Russian). Izv. Vuzov Mat. **1**(20), 152–162 (1961)
167. Shimanov, S.N.: The critical case of a pair of purely imaginary roots for a system with retardation (Russian). Sib. Mat. Zhurn. **2**(3), 467–480 (1961)
168. Shimanov, S.N.: The critical case of a pair of purely imaginary roots for a system with retardation (singular case) (Russian). Sib. Mat. Zhurn. **4**(2), 457–470 (1963)
169. Shnol', E.E.: Nonexistence of algebraic criteria for asymptotic stability under 1:3 resonance (Russian). Preprint no. 45, Inst. Prikl. Mat. Akad. Nauk. SSSR (1977)
170. Shnol', E.E.: Instability of the equilibrium point in autonomous systems of ordinary differential equations (Russian). Preprint of NCBI, Akad. Nauk. SSSR (1983)
171. Shterenlikht, L.M.: Lyapunov canonic transformations and normal Hamiltonian forms. J. Appl. Math. Mech. **39**(4), 578–587 (1975). Translated from Prikl. Mat. Mekh. **39**, 604–613 (1975)
172. Siegel, C.L.: Über die Existenz einer Normalform analytischer Hamiltonischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. Math. Ann. **128**, 144–170 (1954)
173. Siegel, C.L., Moser, J.K.: Lectures on Celestial Mechanics. Springer, New York (1971)
174. Smale, S.: On gradient dynamical systems. Ann. Math. **74**, 199–206 (1961)
175. Sokol'skiy, A.G.: On the stability of an autonomous Hamiltonian system with two degrees in the case of equal frequencies. J. Appl. Math. Mech. **38**(5), 741–749 (1974). Translated from: Prikl. Mat. Mekh. **38**(5), 791–799 (1974)
176. Sokol'skiy, A.G.: On stability of an autonomous Hamiltonian system with two degrees of freedom under first-order resonance. J. Appl. Math. Mech. **41**(1), 20–28 (1977). Translated from Prikl. Mat. Mekh. **41**(1), 24–33 (1977)
177. Sokol'skiy, A.G.: On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of zero frequencies. J. Appl. Math. Mech. **45**(3), 441–449 (1981). Translated from Prikl. Mat. Mekh. **45**(3), 441–446 (1981)
178. Sokol'skiy, A.G.: On the stability of Hamiltonian systems in the case of zero frequencies (Russian). Differ. Uravn. **17**(8), 1509–1510 (1981). Zbl 0467.34043
179. Spring, F., Waldvogel, J.: Chaos in coorbital motion. In: Roy, A.E. (ed.) Predictability, Stability and Chaos in N-Body Systems, pp. 395–410. Plenum, New York (1991)
180. Svirezhev, Y.M., Logofet, D.O.: Stability of Biological Populations (Russian). Nauka, Moscow (1978). MR 537932
181. Tabor, M., Weiss, J.: Analytic structure of the Lorenz system. Phys. Rev. A **24**(4), 2157–2167 (1981)

182. Taliaferro, S.D.: Instability of an equilibrium in a potential field. Arch. Rat. Mech. Anal. **109**, (2), 183–194 (1990)
183. Tamm, I.E.: Fundamentals of the Theory of Electricity (Russian). Nauka, Moscow (1966)
184. Treshchev, D.V.: An estimate of irremovable nonconstant terms in the reducibility problem. In: Kozlov, V.V. (ed.) Dynamical Systems in Classical Mechanics, Transl. Ser. 2, vol. 168, pp. 91–128. American Mathematical Society, Providence (1995)
185. Trofimov, V.V., Fomenko, A.T.: The Algebra and Geometry of Integrable Hamiltonian Differential Equations (Russian). Faktorial, Moscow (1995). MR 1469742
186. Tyson, J.J.: On the appearance of chaos in a model of the Belousov-Zhabotinsky reaction. J. Math. Biol. **5**(4), 351–362 (1978)
187. van der Waerden, B.L.: Algebra, vols. I, II. Springer, New York (1991)
188. Veretennikov, V.G.: Stability and Oscillations of Nonlinear Systems (Russian). Nauka, Moscow (1984). MR 752900
189. Volterra, V.: Leçons sur la théorie mathématique de la lutte pour la vie. Éditions Jacques Gabay, Sceaux (1990). MR 1189803
190. Vuyichich, V.A, Kozlov, V.V.: Lyapunov stability with respect to given state functions. J. Appl. Math. Mech. **55**(4), 442–445 (1991). Translated from: Prikl. Mat. Mekh. **55**(4), 555–559 (1991). Zbl 0786.70015
191. Wallace A.H.: Differential Topology. Initial Steps. Benjamin, New York (1968)
192. Wasow, W.: Asymptotic Expansions for Ordinary Differential Equations. Wiley-Interscience, New York (1965)
193. Whittaker, E.T.: A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Dover, New York (1944)
194. Wiener, Z.: Instability with two zero frequencies. J. Differ. Equ. **103**, 58–67 (1993)
195. Williamson, J.: On an algebraic problem concerning the normal forms of linear dynamical systems. Am. J. Math. **58**, 141–163 (1936)
196. Wintner, A.: The Analytical Foundations of Celestial Mechanics. Princeton University Press, Princeton (1941)
197. Yoshida, H.: Necessary condition for the existence of algebraic first integrals I, II. Celest. Mech. **31**, 363–379, 381–399 (1983)
198. Yoshida, H.: A criterion of the non-existence of an additional integral in Hamiltonian systems with a homogeneous potential. Phys. **29D**, 128–142 (1987)
199. Yoshida, H.: A note on Kowalevsky exponents and the nonexistence of an additional analytic integral. Celest. Mech. **44**, 313–316 (1988)
200. Yoshida, H.: A criterion of the non-existence of an additional analytic integral in Hamiltonian systems with n degrees of freedom. Phys. Lett. A **141**, 108–112 (1989)
201. Zehnder, E.: Generalized implicit function theorems with application to some small divisor problems. Commun. Pure Appl. Math. **28**, 91–140 (1975)
202. Ziglin, S.L.: Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics I,II (Russian). Funkts. Anal. Prilozh. **16**(3), 30–41 (1982); **18**(1), 8–23 (1983). MR 674006, MR 695092
203. Zubov, V.I.: Stability of Motion (Russian). Vyshaya Shkola, Moscow (1973). MR 1445277

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